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Boundary three-point function on AdS_2 D-branes

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Abstract: Using the H_3^+-Liouville relation, I explicitly compute the boundary three-point function on AdS_2 D-branes in H_3^+, and check that it exhibits the expected symmetry properties and has the correct geometrical limit. I then find a simple relation between this boundary three-point function and certain fusing matrix elements, which suggests a formal correspondence between the AdS_2 D-branes and discrete representations of the symmetry group. Concluding speculations deal with the fuzzy geometry of AdS_2 D-branes, strings in the Minkowskian AdS_3, and the hypothetical existence of new D-branes in H_3^+.

Keywords: Conformal Field Models in String Theory, D-branes.

Contents

1. Introduction

In a recent article [1], Hosomichi and I solved the H_3^+ model on a disc with boundary conditions corresponding to AdS_2 D-branes. However the solution was formulated in terms of variables which are well-adapted to the H_3^+-Liouville relation, but which obscure the symmetry of the model. For the structure and consequences of the solution to be understood, the symmetry should be made manifest, and this requires some more work. It is particularly important to perform this work in the case of the boundary three-point function because, coming after the bulk three-point function [2] and bulk-boundary two-point function [1], this completes a set of correlation functions from which all others can be obtained. In addition, the boundary three-point function describes the dynamics of boundary condition changing operators, and makes it possible to investigate the structural properties of the model.

The first purpose of the present article is therefore to explicitly write and analyze the boundary three-point function. This will confirm the correctness of the solution of the H_3^+ model on the disc. In particular, the geometrical ("minisuperspace") analysis and the analysis of the symmetries of the boundary three-point function can be understood as further pieces of evidence for the solution proposed in [1]. The second main purpose of the article is to initiate the study of the structure of the boundary H_3^+ model, with the eventual aim of confronting it with general ideas on the structure of boundary conformal field theories. Of course I cannot a priori assume a general result like the relation between fusing matrix and boundary three-point function to hold in the H_3^+ model, because this non-rational, non-unitary, and non-holomorphically factorizable model violates the assumptions under which such a result is derived. It will however turns out that the boundary three-point function in H_3^+ can indeed be expressed in terms of certain fusing matrix elements, provided one introduces a correspondence between the AdS_2 D-branes and the discrete representations of the symmetry group, although such representations are absent from the spectrum.

The calculation of the relevant H_3^+ fusing matrix elements will not rely on a systematic analysis of the H_3^+ conformal blocks, which is postponed to future work. Rather, I will make a straightforward and somewhat naive use of the H_3^+-Liouville relation, which in certain cases yields the H_3^+ fusing matrix elements in terms of Liouville theory fusing matrix elements. Such an approach is justified a posteriori by the relation with the boundary three-point function.

The plan of the article is as follows. Section 2 is devoted to defining the boundary three-point function (2.11) and deriving some features which can be predicted without knowledge of the exact solution, either from a geometrical calculation or from the analysis of the symmetry of the model. In particular, given the symmetry, the three-point function is parametrized by two structure constants $C_\pm$ (2.14). In section 3, I will use the exact solution [1] for checking these predictions, and give an explicit formula (3.20) for the structure constants. Section 4 is devoted to the computation of fusing matrix elements in H_3^+, and to their relation (4.30) with the boundary three-point function. This will require the formal introduction of discrete representations. The concluding section 5 will offer some speculations which are inspired by these results.

This article can be thought of as a follow-up of [1], which is briefly summarized in [3]. Nevertheless, the necessary results on the H_3^+ model on a disc [4, 1] will be recalled, although not explained in detail. The necessary results on Liouville theory, which come from the works [5 – 8], will also be recalled, mostly in the conventions of the short review [9].

2. The three-point function: predictions

2.1 Geometrical description

The aim of this subsection is to predict the geometrical limit of the boundary three-point function in H_3^+. I will first recall (from [4]) which model is obtained as the geometrical limit of the H_3^+ model, and which quantities should have well-defined limits. This will lead to the definition of a geometrical three-point function, which will then be explicitly computed.

Geometry of H_3^+ and of the AdS_2 D-branes. The three-dimensional Euclidean space H_3^+ can be defined as the set of two-by-two Hermitian matrices h of determinant one, and parametrized by three coordinates $(\phi, \gamma, \bar{\gamma})$ such that $h = \begin{pmatrix} e^{\phi} & e^{\phi}\bar{\gamma} \\ e^{\phi}\gamma & e^{\phi}\gamma\bar{\gamma}+e^{-\phi} \end{pmatrix}$. The space H_3^+ can also be seen as the right coset $SL(2,\mathbb{C})/\,\mathrm{SU}(2)$, on which an $SL(2,\mathbb{C})$ symmetry group acts by left multiplication; the resulting action of $g \in SL(2,\mathbb{C})$ on the Hermitian matrix h is $g \cdot h = ghg^\dagger$. The D-branes of interest are Euclidean AdS_2 branes, which should more accurately be called H_2^+ branes. They are defined by equations of the type $\mathrm{Tr}\ \Omega h = 2\sinh r$ where the real parameter r determines the curvature of H_2^+ while the Hermitian matrix Ω determines its orientation. Such a D-brane intersects the $\phi = \infty$ boundary of H_3^+, which is a two-sphere S^2, and the intersection is a great circle, with an equation of the type $|\gamma - \gamma_0| = R_0$ or $\Re(\mu_0\gamma) = \lambda_0$.

Let me fix the orientation of the AdS_2 branes, and consider only D-branes with the same matrix $\Omega = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, the same great circle at infinity $\gamma + \bar{\gamma} = 0$, and the same preserved $SL(2,\mathbb{R})$ subgroup $\left\{g = \left(\begin{smallmatrix} a & ic \\ -ib & d \end{smallmatrix}\right),\ ad - bc = 1,\ a, b, c, d \in \mathbb{R}\right\}$ of the $SL(2,\mathbb{C})$ symmetry group. This assumption ensures that the theory of open strings stretched between two such D-branes enjoys a maximal amount of symmetry. A further assumption is needed for the theory of open strings on AdS_2 branes to have a geometrical description: open strings should reduce to point particles, which is only possible if they have both ends on the same D-brane. In this subsection I will therefore assume all involved AdS_2 branes to have the same parameter r, thus the same equation $e^{\phi}(\gamma + \bar{\gamma}) = 2\sinh r$. The theory of open strings on this D-brane then has a well-defined geometrical description in the *minisuperspace limit*, as the quantum mechanics of a point particle in AdS_2.

Point particles in AdS_2. Point particles in the Euclidean AdS_2 are described by their wavefunctions: complex-valued functions on AdS_2. Their spectrum, namely the space of such functions, can be organized according to the action of the $SL(2,\mathbb{R})$ symmetry group. Namely, the spectrum is generated by functions

$$\Psi^{\ell}(t|h) = \left(|\gamma + it|^2 e^{\phi} + e^{-\phi}\right)^{\ell}, \tag{2.1}$$

which belong to continuous representations of $SL(2,\mathbb{R})$ of spins $\ell \in -\frac{1}{2} + i\mathbb{R}$ and Casimir eigenvalues $-\ell(\ell+1)$, and $t \in \mathbb{R}$ is the isospin variable. The transformation of such functions under the action of $g \in SL(2,\mathbb{R})$ is indeed

$$\Psi^{\ell}(t|g \cdot h) = |ct - d|^{2\ell}\Psi^{\ell}(g \cdot t|h), \qquad g \cdot t = \frac{at - b}{-ct + d}. \tag{2.2}$$

Let me define the geometrical three-point function on an AdS_2 brane of parameter r as

$$\Omega_3^{\mathrm{geom}} \equiv \int dh\ \delta(e^{\phi}(\gamma + \bar{\gamma}) - 2\sinh r) \prod_{i=1}^{3} \Psi^{\ell_i}(t_i|h), \tag{2.3}$$

where $dh = e^{2\phi} d\phi\ d^2\gamma$ is the $SL(2,\mathbb{C})$-invariant measure on H_3^+. The purpose of this subsection is to obtain the explicit expression of Ω_3^{geom}.

Calculation of $\Omega_3^{\rm geom}$. The calculation goes as follows (neglecting numerical factors). Perform the integral over $\gamma+\bar{\gamma}$ and write $\gamma = e^{-\phi}\sinh r - i\rho$ with $\rho\in\mathbb{R}$, then perform the shift $\phi\to\phi+\log\cosh r$. This yields

$$\Omega_3^{\rm geom} = (\cosh r)^{\sum \ell_i+2}\int e^{\phi}d\phi\ d\rho\ \prod_{i=1}^{3}\left(|\rho-t_i|^2e^{\phi}+e^{-\phi}\right)^{\ell_i} . \tag{2.4}$$

Having made the r-dependence explicit, the next step is to make the t_i-dependence explicit:

$$\Omega_3^{\rm geom} = (\cosh r)^{\sum \ell_i+2}\ |t_{12}|^{\ell_{12}^3}|t_{13}|^{\ell_{13}^2}|t_{23}|^{\ell_{23}^1}\ C^{\rm geom}(\ell_1,\ell_2,\ell_3)\,, \tag{2.5}$$

with the notations $t_{12}=t_1-t_2$ and $\ell_{12}^3=\ell_1+\ell_2-\ell_3$. This formula can be derived by using the $SL(2,\mathbb{R})$ symmetry of $\Omega_3^{\rm geom}$, and its explicit expression in the limit $t_3\to\infty$, after performing the change of variables $(\phi,\rho)\to(\phi-\log|t_{12}|, t_{21}\rho+t_1)$. This also provides the integral expression of $C^{\rm geom}$, the geometrical limit of the three-point structure constant at $r=0$:

$$C^{\rm geom} = \int e^{\phi}d\phi\ d\rho\ \left(\rho^2e^{\phi}+e^{-\phi}\right)^{\ell_1}\left((\rho-1)^2e^{\phi}+e^{-\phi}\right)^{\ell_2}e^{\ell_3\phi} . \tag{2.6}$$

Now introduce variables $(x_1,x_2)=(e^{\phi}\rho, e^{\phi}(1-\rho))$, while allowing e^{ϕ} to take all real values,

$$C^{\rm geom} = \int_{\mathbb{R}^2} dx_1\ dx_2\ |x_1+x_2|^{-\ell_{12}^3-1}(1+x_1^2)^{\ell_1}(1+x_2^2)^{\ell_2} . \tag{2.7}$$

Inserting $1=\int dy\ \delta(y+x_1+x_2)$ and $\delta(y+x_1+x_2)=\int d\theta\ e^{i\theta(y+x_1+x_2)}$ yields

$$\begin{aligned} C^{\rm geom} &= \int d\theta\int dy\ dx_1\ dx_2\ e^{i\theta(y+x_1+x_2)}|y|^{-\ell_{12}^3-1}(1+x_1^2)^{\ell_1}(1+x_2^2)^{\ell_2} \qquad (2.8)\\ &= 2^{\ell_1+\ell_2}\frac{\Gamma(-\ell_{12}^3)\sin\frac{\pi}{2}\ell_{12}^3}{\Gamma(-\ell_1)\Gamma(-\ell_2)}\int_0^\infty d\theta\ \ \theta^{-\ell_3-1}K_{-\ell_1-\frac12}(\theta)K_{-\ell_2-\frac12}(\theta)\,,\\ C^{\rm geom} &= \Gamma(-\tfrac12(\ell_{123}+1))\frac{\Gamma(-\frac12\ell_{12}^3)\Gamma(-\frac12\ell_{13}^2)\Gamma(-\frac12\ell_{23}^1)}{\Gamma(-\ell_1)\Gamma(-\ell_2)\Gamma(-\ell_3)}\,, \qquad (2.9)\end{aligned}$$

where I used standard formulas [10] for the Bessel function with imaginary argument K, and the integral formula (A.8). (And a new notation: $\ell_{123}=\ell_1+\ell_2+\ell_3$.)

The formula for $C^{\rm geom}$ is permutation-symmetric, which is a basic check of its correctness. It vanishes for discrete spins $\ell\in\mathbb{N}$, which explains the absence of discrete representations in the spectrum, in spite of their appearance in tensor products of continuous representations. And it will be shown to agree with the geometrical limit of the exact open string three-point function in subsection 3.3.

2.2 Symmetry

Let me leave the geometrical limit and consider more general boundary three-point functions, where open strings can have their ends on different AdS_2 D-branes. I will now derive the constraints on the boundary three-point function which follow from the assumed symmetries of the model. The symmetry group of the model is an infinite-dimensional loop group, whose Lie algebra is the affine Lie algebra $\widehat{s\ell_2}$. The three-dimensional horizontal subgroup will be most relevant in the following.

Action of the symmetry on the open strings. The global structure of the horizontal subgroup of the symmetry group of the H_3^+ model on the disc was understood only recently [1], because it differs from the $SL(2,\mathbb{R})$ group which is present in the geometrical limit, and which had naively been expected to be present in the general case as well. The correct symmetry group is actually $\widetilde{SL}(2,\mathbb{R})$, the universal covering group, whose elements are pairs $(g,[T])$ with $g = \left(\begin{smallmatrix} a & ic \\ -ib & d \end{smallmatrix}\right)$ an element of the same $SL(2,\mathbb{R})$ subgroup of $SL(2,\mathbb{C})$ as before, and $[T] \in \mathbb{Z}$ an integer. The group multiplication law is $(g,[T]) \cdot (g',[T']) = (gg',[T]+[T']+[g,g'])$ where $[g,g'] \in \{0,1\}$ is the integer part of $T(g)+T(g')$, with $T(g) \in [0,1[$ a timelike coordinate on $SL(2,\mathbb{R})$. (The elements of the additive group $\mathbb{R}$ can similarly be viewed as pairs of an element of $[0,1[$ and an integer, whose addition law would then be similar to the present $\widetilde{SL}(2,\mathbb{R})$ multiplication law.) The action of $\widetilde{SL}(2,\mathbb{R})$ on vertex operators is[1]

$$(g,[T]) \cdot {}_r\Psi^\ell(t|w)_{r'} = |ct-d|^{2\ell} e^{-(k-2)(r-r')\left([T]+\frac{1}{2}+\frac{1}{2}\mathrm{sgn}(t-\frac{d}{c})\right)} {}_r\Psi^\ell(g\cdot t|w)_{r'} \,, \qquad (2.10)$$

where the vertex operator ${}_r\Psi^\ell(t|w)_{r'}$, whose position on the boundary of the worldsheet is $w \in \mathbb{R}$, describes an open string stretched between two AdS_2 branes with the same orientation and parameters r and r'; and $k > 2$ is the level of the H_3^+ model, which is related to the central charge by $c = \frac{3k}{k-2}$, and will sometimes be replaced with the equivalent parameter $b^2 = \frac{1}{k-2}$. Like in the geometrical limit, the spectrum is purely continuous with spins $\ell \in -\frac{1}{2} + i\mathbb{R}$.

Definition of the boundary three-point function. The boundary three-point function is defined as the expectation value

$$\Omega_3 = \left\langle {}_{r_{31}}\Psi^{\ell_1}(t_1|w_1)_{r_{12}} \Psi^{\ell_2}(t_2|w_2)_{r_{23}} \Psi^{\ell_3}(t_3|w_3)_{r_{31}} \right\rangle \,. \qquad (2.11)$$

From the point of view of two-dimensional conformal field theory, this describes the insertion of three vertex operators on the circular boundary of a disc worldsheet. From the target space point of view, this describes three open strings stretched between three AdS_2 branes of parameters r_{12}, r_{23}, r_{13}, whose identical orientation means they coincide at infinity. (For convenience, only two dimensions of H_3^+ are represented here, and the sphere S^2 at infinity is represented as a dashed circle. The open string states are represented as well-localized wiggly lines, although in reality the operators Ψ^{ℓ_i} rather correspond to momentum eigenstates.)

[1] The present convention for the sign of the exponent differs from [1]. The present convention will be consistent with the chosen conventions in Liouville theory through the H_3^+-Liouville relation. I believe that the conventions in [1] were not consistent in this respect.

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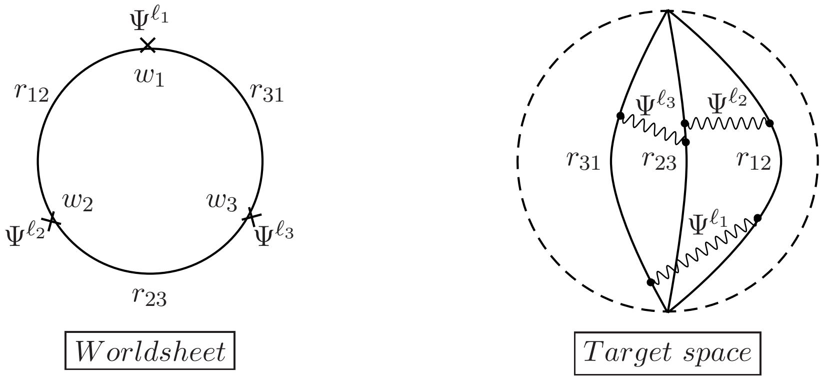

The dependence of the three-point function on the boundary coordinates $w_i \in \mathbb{R}$ is determined by conformal symmetry to be a factor $|w_{12}|^{\Delta_{\ell_3}-\Delta_{\ell_1}-\Delta_{\ell_2}}|w_{23}|^{\Delta_{\ell_1}-\Delta_{\ell_2}-\Delta_{\ell_3}}|w_{13}|^{\Delta_{\ell_2}-\Delta_{\ell_1}-\Delta_{\ell_3}}$, which will be omitted henceforth. Here $\Delta_\ell = -\frac{\ell(\ell+1)}{k-2}$ is the conformal weight of Ψ^ℓ, and $w_{12} = w_1 - w_2$. It is however necessary to keep track of the order of the fields on the boundary of the disc. The three-point function is indeed expected to be invariant under cyclic permutations, but not under a permutation of two fields. This differs from the full permutation symmetry of the boundary three-point function of say Liouville theory. This is because the H_3^+ boundary field ${}_r\Psi^\ell(t|w)_{r'}$ and its symmetry transformation (2.10) are nontrivially affected by the exchange of the two boundary conditions r, r'. In other words, the boundary theory is not invariant under worldsheet parity. Here I am assuming the boundary to be oriented counterclockwise, and the boundary operators to come in the order $1, 2, 3$ like in formula (2.11).

Solving the $\widetilde{SL}(2,\mathbb{R})$ symmetry condition. The $\widetilde{SL}(2,\mathbb{R})$ symmetry condition on the boundary three-point function is

$$\left\langle (g,[T])\cdot\Psi^{\ell_1}\ (g,[T])\cdot\Psi^{\ell_2}\ (g,[T])\cdot\Psi^{\ell_3}\right\rangle = \left\langle \Psi^{\ell_1}\ \Psi^{\ell_2}\ \Psi^{\ell_3}\right\rangle , \tag{2.12}$$

which explicitly reads

$$\begin{aligned}&\Omega_3\left(\frac{at_1-b}{-ct_1+d},\frac{at_2-b}{-ct_2+d},\frac{at_3-b}{-ct_3+d}\right) \qquad (2.13)\\ &\quad = e^{-\frac{k-2}{2}\left[r_{12}(\mathrm{sgn}(t_1-\frac{d}{c})-\mathrm{sgn}(t_2-\frac{d}{c}))+r_{23}(\mathrm{sgn}(t_2-\frac{d}{c})-\mathrm{sgn}(t_3-\frac{d}{c}))+r_{31}(\mathrm{sgn}(t_3-\frac{d}{c})-\mathrm{sgn}(t_1-\frac{d}{c}))\right]}\ \Omega_3\ .\end{aligned}$$

The general solution is found with the help of the identity (A.9),

$$\Omega_3 = |t_{12}|^{\ell_{12}^3}|t_{13}|^{\ell_{13}^2}|t_{23}|^{\ell_{23}^1}e^{\frac{k-2}{2}[r_{12}\mathrm{sgn}t_{12}+r_{23}\mathrm{sgn}t_{23}+r_{31}\mathrm{sgn}t_{31}]}\ C_{\mathrm{sgn}t_{12}t_{23}t_{31}}\ , \tag{2.14}$$

where C_λ is an arbitrary function of the $\widetilde{SL}(2,\mathbb{R})$-invariant combination $\lambda = \mathrm{sgn}t_{12}t_{23}t_{31} = \pm$. Thus, the boundary three-point function is written in terms of two independent structure constants $C_\pm$. This reflects the fact that the tensor product of two continuous representation contains two copies of each continuous representation.

Notice that $r_{12}, r_{23}, r_{31}, C_\pm$ cannot be unambiguously determined from Ω_3. The ambiguity corresponds to the invariance of Ω_3 under $r_{ij} \to r_{ij} + r_0,\ C_\lambda \to e^{\frac{k-2}{2}r_0\lambda}C_\lambda$, which follows from the identity (A.9). This ambiguity will be relevant in the comparison between the exact three-point function and the geometrical prediction.

z	∞		0		1		∞		0		1		∞
sgn(ν_1,ν_2,ν_3)		(−+)		(+-+)		(+−)		(++-)		(-+-)		(-++)	
Notation		[+312]		[-231]		[+123]		[-312]		[+231]		[-123]	

Table 1: The space of values of ν_i as a double cover of the z-axis. (The notation for a regime indicates first $\text{sgn}\nu_1\nu_2\nu_3 = \pm$, then the order of the fields on the worldsheet boundary, starting with the index i such that $\text{sgn}\nu_i = \text{sgn}\nu_1\nu_2\nu_3$).

2.3 Fourier transformation to the ν-basis

The first aim of the next section will be to check that the H_3^+ boundary three-point function predicted by the H_3^+-Liouville relation is of the form (2.14) dictated by the $SL(2,\mathbb{R})$ symmetry. However, the H_3^+-Liouville relation will not directly yield the boundary three-point function Ω_3 of the t-basis fields ${}_r\Psi^\ell(t|w)_{r'}$ used so far, but rather the following ν-basis boundary three-point function

$$\tilde{\Omega}_3 = \prod_{i=1}^{3}\left(|\nu_i|^{\ell_i+1}\int_{\mathbb{R}} dt_i\ e^{i\nu_i t_i}\right)\Omega_3 = \left\langle {}_{r_{31}}\Psi^{\ell_1}(\nu_1|w_1)_{r_{12}}\Psi^{\ell_2}(\nu_2|w_2)_{r_{23}}\Psi^{\ell_3}(\nu_3|w_3)_{r_{31}}\right\rangle \quad (2.15)$$

where the ν-basis boundary fields are defined as

$$ {}_r\Psi^\ell(\nu|w)_{r'} = |\nu|^{\ell+1}\int_{\mathbb{R}} dt\ e^{i\nu t}\ {}_r\Psi^\ell(t|w)_{r'}\ , \qquad \nu\in\mathbb{R}\ . \quad (2.16)$$

The present subsection is therefore devoted to the technical task of computing $\tilde{\Omega}_3$ by straightforward Fourier transformation of the t-basis result (2.14), which amounts to formulating the $\widetilde{SL}(2,\mathbb{R})$ symmetry constraint in the ν-basis.

Properties of the ν-basis. Only two of the three independent $\widetilde{SL}(2,\mathbb{R})$ symmetries have a simple action on ν-basis fields. The first one is t-translation symmetry, which implies ν conservation, so that the ν-basis three-point function $\tilde{\Omega}_3$ must have a $\delta(\nu_1+\nu_2+\nu_3)$ factor. The second one is t-dilatation symmetry, which corresponds to ν-dilatation symmetry, and implies that $\tilde{\Omega}_3$ is a nontrivial function of only one dilatation-invariant real variable, say $z = -\frac{\nu_1}{\nu_2}\in\mathbb{R}$. Note however that only positive dilatations are allowed, namely $\nu_i\to\alpha\nu_i$ with $\alpha>0$. The nontriviality of the transformation $\nu_i\to-\nu_i$ implies that $\tilde{\Omega}_3$ should be thought of as a function on a double cover of $\mathbb{R}$, see table 1.

Let me describe more precisely the ν-dependence of $\tilde{\Omega}_3$. As will follow from the direct calculation of $\tilde{\Omega}_3$, and could alternatively be derived from the local $s\ell(2,\mathbb{R})$ symmetry, $\tilde{\Omega}_3$ is a linear combination of hypergeometric functions of the type:

$$\begin{aligned}\mathcal{F}_\eta^{(3)} &\equiv \delta(\textstyle\sum\nu_i)|\nu_1|^{-\ell_1-\ell_3^\eta-1}|\nu_2|^{\ell_2+1}|\nu_3|^{\ell_3^\eta+1}F\left(\ell_{123^\eta}+2,\ell^1_{23^\eta}+1,2\ell_3^\eta+2,-\dfrac{\nu_3}{\nu_1}\right)\\ &= \delta(\textstyle\sum\nu_i)|\nu_1|^{\ell_1+1}|\nu_2|^{-\ell_2-\ell_3^\eta-1}|\nu_3|^{\ell_3^\eta+1}F\left(\ell_{123^\eta}+2,\ell^2_{13^\eta}+1,2\ell_3^\eta+2,-\dfrac{\nu_3}{\nu_2}\right)\end{aligned}$$

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$$\mathcal{F}^{(2)}_\eta \equiv \delta(\textstyle\sum\nu_i)|\nu_1|^{\ell_1+1}|\nu_2|^{\ell_2^\eta+1}|\nu_3|^{-\ell_1-\ell_2^\eta-1}F\left(\ell_{12^\eta 3}+2,\ell^3_{12^\eta}+1,2\ell_2^\eta+2,-\frac{\nu_2}{\nu_3}\right) \quad (2.17)$$
$$= \delta(\textstyle\sum\nu_i)|\nu_1|^{-\ell_2^\eta-\ell_3-1}|\nu_2|^{\ell_2^\eta+1}|\nu_3|^{\ell_3+1}F\left(\ell_{12^\eta 3}+2,\ell^1_{2^\eta 3}+1,2\ell_2^\eta+2,-\frac{\nu_2}{\nu_1}\right)$$
$$\mathcal{F}^{(1)}_\eta \equiv \delta(\textstyle\sum\nu_i)|\nu_1|^{\ell_1^\eta+1}|\nu_2|^{-\ell_1^\eta-\ell_3-1}|\nu_3|^{\ell_3+1}F\left(\ell_{1^\eta 23}+2,\ell^2_{1^\eta 3}+1,2\ell_1^\eta+2,-\frac{\nu_1}{\nu_2}\right)$$
$$= \delta(\textstyle\sum\nu_i)|\nu_1|^{\ell_1^\eta+1}|\nu_2|^{\ell_2+1}|\nu_3|^{-\ell_1^\eta-\ell_2-1}F\left(\ell_{1^\eta 23}+2,\ell^3_{1^\eta 2}+1,2\ell_1^\eta+2,-\frac{\nu_1}{\nu_3}\right)$$

where $\eta=\pm$ and $\ell^+=\ell, \ell^-=-\ell-1$ thus $\ell^3_{12^\eta}=\ell_1+\ell_2^\eta-\ell_3$. The arguments of the hypergeometric functions are assumed to belong to $]-\infty,1[$, which happens for $\mathcal{F}^{(3)}_\eta$ provided $\nu_1\nu_2<0$. (In particular, $\mathcal{F}^{(3)}_\eta$ has a power-like behaviour near $\nu_3=0$, but behaves as a linear combination of powers of $|\nu_1|$ and $|\nu_2|$ near $\nu_1=0$ and $\nu_2=0$ respectively.) Therefore, out of the three alternative bases $\mathcal{F}^{(1)}_\eta,\mathcal{F}^{(2)}_\eta,\mathcal{F}^{(3)}_\eta$, only two can be used for given values of ν_1,ν_2,ν_3. For instance, in the regimes $[\pm 312]$, the two bases $\mathcal{F}^{(1)}_\eta,\mathcal{F}^{(2)}_\eta$.

So the ν-basis three-point function $\tilde{\Omega}_3$ should have expressions of the form

$$\tilde{\Omega}_3=\sum_{\lambda=\pm}C_\lambda\sum_{\eta=\pm}T^{[\mathrm{sgn}\nu_i]}_{\lambda,\eta}\mathcal{F}^{(j)}_\eta\,, \quad (2.18)$$

where $[\mathrm{sgn}\nu_i]$ denotes a regime, for instance $[+312]$, and j denotes one of the two allowed bases in that regime, here $j=1,2$. Depending on this choice of basis, the coefficient will be denoted as $T^{[+3(1)2]}_{\lambda,\eta}$ or $T^{[+31(2)]}_{\lambda,\eta}$. These coefficients relate the ν-basis three-point structure constants $\tilde{C}^{[\mathrm{sgn}\nu_i]}_\eta=\sum_{\lambda=\pm}C_\lambda T^{[\mathrm{sgn}\nu_i]}_{\lambda,\eta}$, which depend on the choices of regime and basis, to the t-basis three-point structure constants C_λ, which do not.

Calculation of $\tilde{\Omega}_3$. Let me explicitly demonstrate that $\tilde{\Omega}_3$ indeed has an expression of the form (2.18), and determine the coefficients $T_{\lambda,\eta}$, by computing the integral (2.15). This integral can be split into six terms corresponding to the six possible orderings of t_1,t_2,t_3 on the real line. Up to a global r_{ij}-dependent factor, the ordering $t_1<t_2<t_3$ yields the following term:

$$J_{123}\equiv\prod_{i=1}^3|\nu_i|^{\ell_i+1}\int_{t_1<t_2<t_3}dt_1\,dt_2\,dt_3\,e^{i(\nu_1t_1+\nu_2t_2+\nu_3t_3)}|t_{12}|^{\ell^3_{12}}|t_{23}|^{\ell^1_{23}}|t_{13}|^{\ell^2_{13}}\,. \quad (2.19)$$

Introduce a variable u by $|t_{13}|^{\ell^2_{13}}=\frac{1}{\Gamma(-\ell^2_{13})}\int_0^\infty du\,e^{-u|t_1-t_3|}u^{-\ell^2_{13}-1}$. Shift $t_1\to t_1+t_2$ and $t_3\to t_3+t_2$, then integrate over t_i, and find

$$J_{123}=\delta(\nu_1+\nu_2+\nu_3)\prod_{i=1}^3|\nu_i|^{\ell_i+1} \quad (2.20)$$
$$\times\frac{\Gamma(\ell^3_{12}+1)\Gamma(\ell^1_{23}+1)}{\Gamma(-\ell^2_{13})}\int_0^\infty du\,u^{-\ell^2_{13}-1}(u+i\nu_1)^{-\ell^3_{12}-1}(u-i\nu_3)^{-\ell^1_{23}-1}\,.$$

The result is an hypergeometric function [10], which is a priori ambiguous when its (real) argument belongs to $]1,\infty[$. In this case, by construction, the hypergeometric function is

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determined by analytic continuation from the region $i\nu_1, -i\nu_3 \in \mathbb{R}_+$. This understood, the result can be written as

$$J_{123} = \Gamma(\ell_{12}^3+1)\Gamma(\ell_{23}^1+1)\frac{\Gamma(\ell_{123}+2)}{\Gamma(2\ell_2+2)}e^{i\frac{\pi}{2}(\ell_{123}+2)\mathrm{sgn}\nu_3}\mathcal{F}_+^{(2)} \ . \tag{2.21}$$

Now consider all six terms contributing to the integral (2.15) in the regime $[\sigma 123]$ with $\sigma = \mathrm{sgn}\nu_1 = -\mathrm{sgn}\nu_2 = -\mathrm{sgn}\nu_3$. The four terms $J_{123}, J_{132}, J_{231}, J_{321}$ yield "good" hypergeometric functions $\mathcal{F}_+^{(2)}, \mathcal{F}_+^{(3)}$ with arguments in $]-\infty, 1[$, whereas the two remaining integrals J_{213}, J_{312} yield "bad" hypergeometric functions $\mathcal{F}_+^{(1)}$ with arguments in $]1,\infty[$. These can however be unambiguously rewritten as combinations of either $\mathcal{F}_\pm^{(2)}$ or $\mathcal{F}_\pm^{(3)}$ functions. The end result is $\tilde{\Omega}_3 = \sum_\lambda C_\lambda \sum_\eta T_{\lambda,\eta}^{[\sigma 1(2)3]}\mathcal{F}_\eta^{(2)}$, with the blocks $\mathcal{F}_\eta^{(2)}$ of eq. (**??**) and the coefficients

$$T_{\lambda,+}^{[\sigma 1(2)3]} = \frac{1}{8}\Gamma(\ell_{123}+2)\Gamma(\ell_{23}^1+1)\Gamma(\ell_{12}^3+1)\Gamma(-2\ell_2-1)e^{i\lambda\sigma\frac{\pi}{2}\ell_{123}} \tag{2.22}$$

$$\times\left[e^{\lambda\frac{r_{23}-r_{31}-r_{12}}{2b^2}}\sin\pi\ell_{23}^1 + e^{\lambda\frac{r_{12}-r_{23}-r_{31}}{2b^2}}\sin\pi\ell_{12}^3 - e^{\lambda\frac{r_{31}-r_{12}-r_{23}}{2b^2}}e^{-i\lambda\sigma\pi\ell_{123}}\sin 2\pi\ell_2\right],$$

$$T_{\lambda,-}^{[\sigma 1(2)3]} = -\frac{\pi}{4}\Gamma(\ell_{13}^2+1)\Gamma(2\ell_2+1)e^{-\lambda\frac{r_{31}}{2b^2}}e^{i\lambda\sigma\frac{\pi}{2}(\ell_{13}^2+1)}\sin\left(\pi\ell_2 + i\sigma\frac{r_{23}-r_{12}}{2b^2}\right) \ . \tag{2.23}$$

This completes the computation of the Fourier transform $\tilde{\Omega}_3$ of the general solution Ω_3 (2.14) of the $\widetilde{SL}(2,\mathbb{R})$ symmetry condition. The coefficients $T_{\lambda,\eta}$ which appear in the result will play an important role in the following, so let me study some of their properties.

Some properties of the coefficients $T_{\lambda,\eta}$. The determinant of the 2×2 matrix $T_{\lambda,\eta}$ is

$$\det T^{[\sigma 1(2)3]} = \frac{i\pi^2\sigma}{8(2\ell_2+1)}\Gamma(\ell_{12}^3+1)\Gamma(\ell_{13}^2+1)\Gamma(\ell_{23}^1+1)\Gamma(\ell_{123}+2) \tag{2.24}$$

$$\times\sin\left(\pi\ell_1 + i\sigma\frac{r_{31}-r_{12}}{2b^2}\right)\sin\left(\pi\ell_2 + i\sigma\frac{r_{23}-r_{12}}{2b^2}\right)\sin\left(\pi\ell_3 + i\sigma\frac{r_{31}-r_{23}}{2b^2}\right) ,$$

and its inverse $\left(T^{-1}\right)_{\lambda,\eta} = \frac{\eta\lambda}{\det T}T_{-\eta,-\lambda}$.

The existence of the two bases $\mathcal{F}_\pm^{(2)}$ and $\mathcal{F}_\pm^{(3)}$ means $\sum_\eta T_{\lambda,\eta}^{[\sigma 1(2)3]}\mathcal{F}_\eta^{(2)} = \sum_{\eta'} T_{\lambda,\eta'}^{[\sigma 12(3)]}\mathcal{F}_{\eta'}^{(3)}$. Given the relations $\sum_\eta \mathcal{F}_\eta^{(i)} M_{\eta\eta'}^{(ij)k} = \mathcal{F}_{\eta'}^{(j)}$ between the two bases of conformal blocks $\mathcal{F}^{(i)}, \mathcal{F}^{(j)}$ in regimes where $\mathrm{sgn}\nu_i = \mathrm{sgn}\nu_j$, this implies relations of the type

$$T_{\lambda,\eta}^{[\sigma 1(2)3]} = \sum_{\eta'} M_{\eta\eta'}^{(23)1} T_{\lambda,\eta'}^{[\sigma 12(3)]} , \tag{2.25}$$

where the monodromy matrix is

$$M_{\eta\eta'}^{(23)1} = \frac{\Gamma(2\ell_3^{\eta'}+2)\Gamma(-2\ell_2^\eta-1)}{\Gamma(1+\ell_1-\ell_2^\eta+\ell_3^{\eta'})\Gamma(-\ell_1-\ell_2^\eta+\ell_3^{\eta'})} , \quad \eta,\eta' = \pm \ . \tag{2.26}$$

(Such relations can be explicitly checked using $T_{\lambda,\eta}^{[\sigma 12(3)]} = T_{\lambda,\eta}^{[-\sigma 1(3)2]}$.)

$\widetilde{SL}(2,\mathbb{R})$ **symmetry condition in the ν-basis.** Finally, examining the coefficients $T_{\lambda,\eta}$ yields the ν-basis formulation of the $\widetilde{SL}(2,\mathbb{R})$ symmetry condition, that is the formulation which will be used in the next section. The global structure of the symmetry group $\widetilde{SL}(2,\mathbb{R})$ is actually encoded in the behaviour of $\tilde{\Omega}_3$ when each of the ν_i vanish, say $\nu_2 = 0$. Such a point separates two regimes where the $\mathcal{F}^{(2)}_\eta$ basis can be used, say $[\sigma 123]$ and $[-\sigma 312]$. It turns out that the coefficient $T_{\lambda,+}$ is continuous across this singularity, whereas $T_{\lambda,-}$ has a jump:

$$T^{[\sigma 1(2)3]}_{\lambda,+} = T^{[-\sigma 31(2)]}_{\lambda,+} \quad , \quad T^{[\sigma 1(2)3]}_{\lambda,-} = \frac{\sin\left(\pi\ell_2 + i\sigma\frac{r_{23}-r_{12}}{2b^2}\right)}{\sin\left(\pi\ell_2 - i\sigma\frac{r_{23}-r_{12}}{2b^2}\right)} T^{[-\sigma 31(2)]}_{\lambda,-} \ . \tag{2.27}$$

Since this does not depend on λ, this can be interpreted as the jump condition on the ν-basis three-point structure constants $\tilde{C}^{[\mathrm{sgn}\nu_i]}_\eta = \sum_{\lambda=\pm} C_\lambda T^{[\mathrm{sgn}\nu_i]}_{\lambda,\eta}$. Thus, $\widetilde{SL}(2,\mathbb{R})$ symmetry relates the ν-basis structure constants in the six regimes (1). Only two of these structure constants are independent, as is expected from their relation with the two t-basis structure constants C_λ.

3. The three-point function: explicit calculation

The symmetry properties of the three-point function, in other words the kinematics, leave the two structure constants C_λ in (2.14) undetermined. The geometrical calculation only gives very partial information on these structure constants. A full determination requires a more powerful dynamical principle. The principle which I will now use is the relation of the H_3^+ model with Liouville theory [11, 1]. The boundary three-point function following from this principle leads to a crossing-symmetric four-point function [1]. The agreement of the H_3^+-Liouville relation with the $\widetilde{SL}(2,\mathbb{R})$ symmetry analysis and with the geometrical calculation is however not obvious, and will have to be checked explicitly.

3.1 The three-point function from Liouville theory

The H_3^+-Liouville relation predicts all correlators of the H_3^+ model on a disc in terms of correlators of Liouville theory on a disc. In this subsection I will review this prediction in the particular case of the H_3^+ boundary three-point function, and show that in this case the relevant Liouville correlators can be explicitly computed.

Prediction of the boundary three-point function. According to [1],

$$\tilde{\Omega}_3 = \delta(\textstyle\sum\nu_i)|\sum\nu_i w_i|^{1+\frac{3}{2b^3}} |\nu_1\nu_2\nu_3 w_{12}w_{23}w_{31}|^{-\frac{1}{2b^2}} \left\langle B^{\beta_1}(w_1)B^{\beta_2}(w_2)B^{\beta_3}(w_3)B^{-\frac{1}{2b}}(y)\right\rangle \tag{3.1}$$

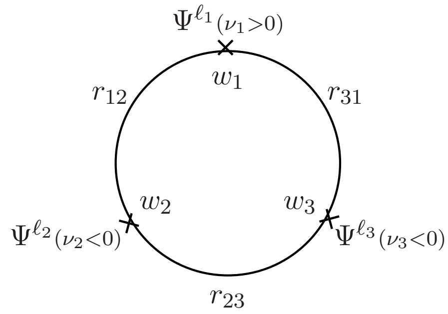
The correlator is a disc boundary four-point function in Liouville theory at central charge $c_L = 1 + 6Q^2$ with $Q = b + b^{-1}$ and $b^2 = \frac{1}{k-2}$, which involves three boundary fields of momenta $\beta_i = b(\ell_i+1) + \frac{1}{2b}$ and conformal weight $\beta_i(Q-\beta_i)$, together with one degenerate boundary field of momentum $-\frac{1}{2b}$, whose position $y = -\frac{\nu_1 w_2 w_3+\nu_2 w_3 w_1+\nu_3 w_1 w_2}{\nu_1 w_1+\nu_2 w_2+\nu_3 w_3}$ is more elegantly defined as

$$\varphi(y) = 0 \qquad \text{where} \qquad \varphi(y) \equiv \sum_i \frac{\nu_i}{y - w_i} \ . \tag{3.2}$$

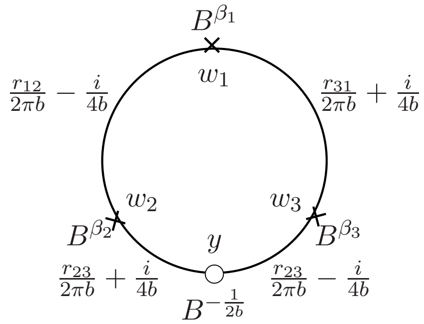
The degenerate field $B^{-\frac{1}{2b}}(y)$ needs not always be located between w_3 and w_1 as in (3.1), but can live at any position on the worldsheed boundary, depending on the variables ν_i: more precisely, between fields at w_i and w_j if and only if $\nu_i\nu_j > 0$. The behaviour of Liouville theory on the boundary of the worldsheet is assumed to be characterized by so-called FZZT branes [5, 6]. The parameter of the FZZT brane at a point w of the boundary is assumed to be[2]

$$s = \frac{r}{2\pi b} - \frac{i}{4b}\mathrm{sgn}\varphi(w)\,, \tag{3.3}$$

where r is the H_3^+ model's boundary parameter (r_{12},r_{23} or r_{31}) at the same point w. In the regime [+123] i.e. $\nu_2, \nu_3 < 0 < \nu_1$ the worldsheet looks like

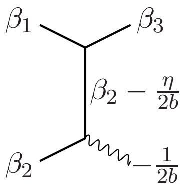
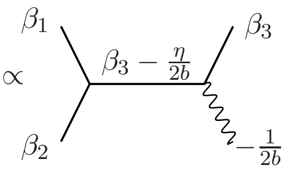


H_3^+ *model*



Liouville theory

Calculation of the relevant Liouville four-point function. Due to the presence of the degenerate field $B^{-\frac{1}{2b}}$, the four-point function in eq. (3.1) obeys a second-order differential equation [12]. The conformal blocks which solve this equation are, up to power factors, hypergeometric functions of cross-ratios of the type $\frac{(y-w_2)(w_1-w_3)}{(y-w_3)(w_1-w_2)} = -\frac{\nu_2}{\nu_3}$. It can be checked[3] that these hypergeometric solutions, combined with the extra factors in eq. (3.1), yield the functions $\mathcal{F}_\eta^{(i)}$ (**??**). (Here and in the following I omit the w-dependence of the H_3^+ three-point function.) The two alternative bases of conformal blocks for given values of sgnν_i correspond to two possible decompositions of the Liouville four-point function. For instance, if $w_2 < y < w_3$ then the field $B^{-\frac{1}{2b}}(y)$ can be associated with either $B^{\beta_2}(w_2)$ or $B^{\beta_3}(w_3)$. In the former case, this means choosing the basis of conformal blocks $\mathcal{F}_\eta^{(2)}$, such that each block $\mathcal{F}_\pm^{(2)}$ has a power-like behaviour in the limit $y \to w_2 \Leftrightarrow \nu_2 \to 0$. This basis has two elements $\eta = \pm$, which correspond to the two fusion channels $B^{-\frac{1}{2b}} \times B^{\beta_2} \to \sum_{\eta=\pm} B^{\beta_2-\frac{\eta}{2b}}$. The corresponding Liouville conformal blocks can be drawn as follows:

$\mathcal{F}_\eta^{(2)} \propto$  , $\mathcal{F}_\eta^{(3)} \propto$  .   (3.4)

[2]The convention for the Liouville boundary parameter s is that the boundary cosmological constant is proportional to $\cosh 2\pi bs$.

[3]A similar calculation was written explicitly in [11] in the case of the relation between the H_3^+ three-point function and the Liouville four-point function *on a sphere*.

The coefficients of the decomposition of the Liouville four-point function in conformal blocks are certain Liouville structure constants. In the regime $[\sigma 1(2)3]$ with the choice of basis $\mathcal{F}^{(2)}_\eta$, the H_3^+ three-point function (3.1) then reads

$$\begin{aligned}\tilde{\Omega}_3 = \sum_{\eta=\pm} C^L &\left(\beta_1 \,\Big|_{\frac{r_{12}}{2\pi b}-\sigma\frac{i}{4b}}\, \beta_2 - \tfrac{\eta}{2b} \,\Big|_{\frac{r_{23}}{2\pi b}-\sigma\frac{i}{4b}}\, \beta_3 \,\Big|_{\frac{r_{31}}{2\pi b}+\sigma\frac{i}{4b}}\right) \\ &\times C^L \left(\beta_2 \,\Big|_{\frac{r_{23}}{2\pi b}+\sigma\frac{i}{4b}}\, -\tfrac{1}{2b} \,\Big|_{\frac{r_{23}}{2\pi b}-\sigma\frac{i}{4b}}\, Q-\beta_2+\tfrac{\eta}{2b} \,\Big|_{\frac{r_{12}}{2\pi b}-\sigma\frac{i}{4b}}\right) \mathcal{F}^{(2)}_\eta \,, \end{aligned} \tag{3.5}$$

where the C^L are Liouville three-point structures constants.

Liouville theory structure constants. The Liouville three-point structure constant is explicitly known [7] as a function of the three momenta β_i and the three boundary parameters s_{ij}:

$$\begin{aligned} C^L\left(\beta_1 \,\Big|_{s_{12}}\, \beta_2 \,\Big|_{s_{23}}\, \beta_3 \,\Big|_{s_{31}}\right) &= \mu_L^{\frac{Q-\beta_{123}}{2b}} \frac{\Gamma_b(2Q-\beta_{123})\Gamma_b(\beta_{23}^1)\Gamma_b(Q-\beta_{13}^2)\Gamma_b(Q-\beta_{12}^3)}{\Gamma_b(Q-2\beta_3)\Gamma_b(Q-2\beta_2)\Gamma_b(Q-2\beta_1)\Gamma_b(Q)} \\ &\times \frac{S_b(Q-\beta_3+is_{31}-is_{23})S_b(Q-\beta_3-is_{23}-is_{31})}{S_b(\beta_2+is_{12}-is_{23})S_b(\beta_2-is_{23}-is_{12})} \\ &\times \frac{1}{i}\int_{Q-i\infty}^{Q+i\infty} dp\, \prod_{i=1}^{4} \frac{S_b(U_i+p)}{S_b(V_i+p)}\,, \end{aligned} \tag{3.6}$$

where the special functions Γ_b and S_b are described in the appendix, μ_L is the renormalized Liouville cosmological constant, and the coefficients U_i, V_i read

$$\begin{array}{ll} U_1 = is_{31} - \beta_1 & V_1 = -is_{23} - \beta_1 + \beta_3 \\ U_2 = -is_{31} - \beta_1 & V_2 = Q - is_{23} - \beta_1 - \beta_3 \\ U_3 = -Q + \beta_2 - is_{23} & V_3 = is_{12} \\ U_4 = -\beta_2 - is_{23} & V_4 = -is_{12} \end{array} \tag{3.7}$$

In this formula the symmetries of C^L are not manifest: neither the invariances under permuations of the indices and under individual reflections of boundary parameters $s_{ij} \to -s_{ij}$, nor the reflection symmetry $C^L\left(\beta_1 \,\Big|_{s_{12}}\, \beta_2 \,\Big|_{s_{23}}\, \beta_3 \,\Big|_{s_{31}}\right) = R^L_{s_{31},s_{12}}(\beta_1)\, C^L\left(Q-\beta_1 \,\Big|_{s_{12}}\, \beta_2 \,\Big|_{s_{23}}\, \beta_3 \,\Big|_{s_{31}}\right)$ (where R^L is given in eq. (3.10)).

The degenerate structure constant $C^L(\beta_2| -\frac{1}{2b}|Q-\beta_2+\frac{\eta}{2b})$ in (3.5) follows from the known formulas [5]

$$C^L\left(\beta \,\Big|_{s}\, -\tfrac{1}{2b} \,\Big|_{s-\sigma\frac{i}{2b}}\, Q-\beta+\tfrac{1}{2b} \,\Big|_{s'}\right) = 1\,, \tag{3.8}$$

$$C^L\left(\beta \,\Big|_{s}\, -\tfrac{1}{2b} \,\Big|_{s-\sigma\frac{i}{2b}}\, Q-\beta-\tfrac{1}{2b} \,\Big|_{s'}\right) = R^L_{s',s}(\beta) R^L_{s-\sigma\frac{i}{2b},s'}(Q-\beta-\tfrac{1}{2b})\,. \tag{3.9}$$

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The first formula is actually a normalization convention, from which the second one is deduced by using the boundary reflection relation ${}_sB^{\beta}_{s'} = R^L_{s,s'}(\beta)\; {}_sB^{Q-\beta}_{s'}$, where the boundary reflection coefficient is

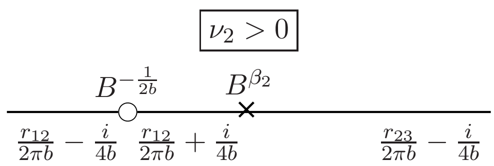
$$R^L_{s,s'}(\beta) = \mu_L^{\frac{Q-2\beta}{2b}} \frac{\Gamma_b(2\beta - Q)}{\Gamma_b(Q-2\beta)} \prod_{\pm,\pm} S_b(Q - \beta \pm is \pm is') \ . \tag{3.10}$$

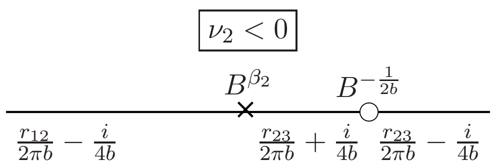
3.2 Check of the symmetry

The formula (3.5) for the ν-basis three-point function $\tilde{\Omega}_3$ is explicit but not particularly illuminating, and it depends on the choices of a particular regime of values of ν_i and of a particular basis of conformal blocks. I will now recast it as a formula for the t-basis structure constants C_λ defined in (2.14), which have no such restrictions and enjoy nicer symmetry properties.

Before doing this, it is however necessary to show that the explicit formula for $\tilde{\Omega}_3$ is indeed compatible with the $\widetilde{SL}(2,\mathbb{R})$ symmetry which underlies the very definition of C_λ. Recall that the $\widetilde{SL}(2,\mathbb{R})$ symmetry condition for the boundary three-point function can be formulated as a condition on its behaviour across a singularity of the type $\nu_2 = 0$, see eq. (2.27). So how does the explicit expression (3.5) behave near $\nu_2 = 0$?

The three-point function $\tilde{\Omega}_3$ near $\nu_2 = 0$. This amounts to studying the behaviour of the Liouville four-point function in (3.1) near $y = w_2$, at which point the degenerate field $B^{-\frac{1}{2b}}(y)$ crosses the field $B^{\beta_2}(w_2)$. Assuming $\nu_1 > 0$ and $\nu_3 < 0$, the worldsheet near w_2 then looks like:





The most complicated factor in (3.5), namely $C^L(\beta_1|\beta_2 - \frac{\eta}{2b}|\beta_3)$, is actually continuous across $\nu_2 = 0$. This factor is indeed a Liouville three-point structure constant involving the field $B^{\beta_2 - \frac{\eta}{2b}}$ which results from the fusion of $B^{-\frac{1}{2b}}$ and B^{β_2}: once they have fused, it does not matter which directions the fields came from. On the other hand, the relative positions of the two fields influence the other factor $C^L(\beta_2| - \frac{1}{2b}|Q - \beta_2 + \frac{\eta}{2b})$ in the case $\eta = -$, because this factor is then sensitive to the boundary parameter between the two fields, as is clear from eq. (3.9):

$$\frac{C^L\left(\beta_2 \underset{\frac{r_{23}}{2\pi b}+\sigma\frac{i}{4b}}{|} -\frac{1}{2b} \underset{\frac{r_{23}}{2\pi b}-\sigma\frac{i}{4b}}{|} Q-\beta_2+\frac{1}{2b} \underset{\frac{r_{12}}{2\pi b}-\sigma\frac{i}{4b}}{|}\right)}{C^L\left(\beta_2 \underset{\frac{r_{23}}{2\pi b}-\sigma\frac{i}{4b}}{|} Q-\beta_2+\frac{1}{2b} \underset{\frac{r_{12}}{2\pi b}-\sigma\frac{i}{4b}}{|} -\frac{1}{2b} \underset{\frac{r_{12}}{2\pi b}+\sigma\frac{i}{4b}}{|}\right)} = \frac{R^L_{\frac{r_{12}}{2\pi b}-\sigma\frac{i}{4b},\frac{r_{23}}{2\pi b}+\sigma\frac{i}{4b}}(\beta_2)}{R^L_{\frac{r_{12}}{2\pi b}+\sigma\frac{i}{4b},\frac{r_{23}}{2\pi b}-\sigma\frac{i}{4b}}(\beta_2)} \tag{3.11}$$

$$= \frac{\sin\left(\pi\ell_2 + i\sigma\frac{r_{23}-r_{12}}{2b^2}\right)}{\sin\left(\pi\ell_2 - i\sigma\frac{r_{23}-r_{12}}{2b^2}\right)} \ .$$

The agreement of this formula with the $\widetilde{SL}(2,\mathbb{R})$ symmetry condition eq. (2.27) demonstrates the consistency of the H_3^+-Liouville relation for the boundary three-point function with the $\widetilde{SL}(2,\mathbb{R})$ symmetry.

Determination of the structure constants C_λ. Let me compare the expression (3.5) of $\tilde{\Omega}_3$ with the expression (2.18) of an $\widetilde{SL}(2,\mathbb{R})$-symmetric three-point function in the ν-basis. Many apparently different expressions for C_λ can be obtained in the different regimes of ν_i, but they are all guaranteed to be equivalent by the $\widetilde{SL}(2,\mathbb{R})$ symmetry. The two regimes [± 123] alone yield four equations for the two unknowns $C_\pm$, schematically

$$C^L_\sigma(\beta_1|\beta_2 - \tfrac{\eta}{2b}|\beta_3) C^L_\sigma(\beta_2| - \tfrac{1}{2b}|Q - \beta_2 + \tfrac{\eta}{2b}) = \sum_{\lambda=\pm} C_\lambda T^{[\sigma 1(2)3]}_{\lambda,\eta}, \quad \forall \sigma = \pm, \eta = \pm\ . \tag{3.12}$$

A relatively simple formula for C_λ is obtained by solving the two equations ($\sigma = \pm, \eta = -$):

$$\begin{aligned} C_\lambda\left(\ell_1 \underset{r_{12}}{|} \ell_2 \underset{r_{23}}{|} \ell_3 \underset{r_{31}}{|}\right) &= -\tfrac{2}{\pi^3}\Gamma(-\ell_{13}^2) R_{r_{12},r_{23}}(\ell_2) \\ &\times \textstyle\sum_{\sigma=\pm} e^{\lambda\left(\frac{r_{31}}{2b^2} - i\sigma\frac{\pi}{2}\ell^2_{13}\right)} C^L\left(\beta_1 \underset{\frac{r_{12}}{2\pi b}-\sigma\frac{i}{4b}}{|} Q - \beta_2 - \tfrac{1}{2b} \underset{\frac{r_{23}}{2\pi b}-\sigma\frac{i}{4b}}{|} \beta_3 \underset{\frac{r_{31}}{2\pi b}+\sigma\frac{i}{4b}}{|}\right) \end{aligned}, \tag{3.13}$$

where the H_3^+ boundary reflection coefficient $R_{r_{12},r_{23}}(\ell_2)$ will shortly be introduced in (3.16), the Liouville boundary three-point function C^L is still given by (3.6), with Liouville momenta still given by $\beta_i = b(\ell_i + 1) + \frac{1}{2b}$. (For a fully explicit formula, see eq. (3.20) below.)

The manifest symmetry of (3.13) under $1 \leftrightarrow 3$ shows that C_λ is invariant not only under cyclic permutations, but under all permutations. Equivalently, the full boundary three-point function Ω_3 (2.14) is invariant under permutations, combined with $t \to -t$ in the case of odd permutations. This invariance of Ω_3 follows from the invariance of the Liouville four-point function (3.1) under cyclic permutations and worldsheet parity.

Reflection properties of the three-point function. For the sake of completeness, and also in order to introduce the useful quantities $R_{r,r'}(\ell)$ and $N^\sigma_{r,r'}(\ell)$, let me discuss the reflection of boundary fields and correlators in H_3^+. By reflection I mean the relation between fields of spins ℓ and $-\ell-1$, which transform in the same representation of $\widetilde{SL}(2,\mathbb{R})$. The reflection of the t-basis boundary field[4] is fairly complicated in that it involves an integral over the isospin variable t,

$${}_r\Psi^\ell(t|w)_{r'} = R_{r,r'}(\ell) \int_{\mathbb{R}} dt'\ |t-t'|^{2\ell} e^{-\frac{k-2}{2}(r-r')\mathrm{sgn}(t-t')}\ {}_r\Psi^{-\ell-1}(t'|w)_{r'}\ , \tag{3.15}$$

with the t-basis reflection number (which is invariant under $r \leftrightarrow r'$)

$$R_{r,r'}(\ell) = N^\sigma_{r,r'}(\ell) R^L_{\frac{r}{2\pi b}+\sigma\frac{i}{4b}, \frac{r'}{2\pi b}-\sigma\frac{i}{4b}}(\beta) \quad \text{with} \quad N^\sigma_{r,r'}(\ell) = \frac{\pi}{\Gamma(2\ell+1)} \frac{1}{\sin(\pi\ell + i\sigma\frac{r-r'}{2b^2})} \tag{3.16}$$

[4]Knowing the reflection behaviour of fields is equivalent to knowing the boundary two-point function [1]

$$\left\langle {}_r\Psi^{\ell_1}(t_1|w_1)_{r'} \Psi^{\ell_2}(t_2|w_2)_r \right\rangle = \delta(\ell_1+\ell_2+1)\delta(t_{12}) + \delta(\ell_1-\ell_2)\tilde{R}^H_{r,r'}(\ell_1)|t_{12}|^{2\ell_1} e^{\frac{1}{2}(k-2)(r-r')\mathrm{sgn}t_{12}}\ . \tag{3.14}$$

where $\beta = b(\ell+1) + \frac{1}{2b}$. The behaviour of C_λ under reflection can in principle be directly deduced from the behaviour of individual boundary fields. It is however simpler to formulate the problem in the ν-basis, which (as follows from the H_3^+-Liouville relation) actually diagonalizes reflection:

$$ {}_r\Psi^\ell(\nu|w)_{r'} = R^L_{\frac{r}{2\pi b}+\frac{i}{4b}\mathrm{sgn}\nu,\frac{r'}{2\pi b}-\frac{i}{4b}\mathrm{sgn}\nu}(\beta) \quad {}_r\Psi^{-\ell-1}(\nu|w)_{r'} \ . \tag{3.17} $$

A third way to deduce the reflection of C_λ is to directly use their expression in terms of the (reflection-friendly) Liouville structure constants (3.12). The result is

$$ C_\lambda\left(\ell_1 \underset{r_{12}}{|} \ell_2 \underset{r_{23}}{|} \ell_3 \underset{r_{31}}{|}\right) = \sum_{\lambda'} R^{(2)}_{\lambda\lambda'}\left(\ell_1 \underset{r_{12}}{|} \ell_2 \underset{r_{23}}{|} \ell_3\right) C_{\lambda'}\left(\ell_1 \underset{r_{12}}{|} -\ell_2-1 \underset{r_{23}}{|} \ell_3 \underset{r_{31}}{|}\right) , \tag{3.18} $$

where the (r_{31}-independent) reflection matrix for the spin ℓ_2 is

$$ \begin{aligned} R^{(2)} &= -\frac{1}{2\pi^2}\Gamma(2\ell_2+1)\Gamma(-\ell_{12}^3)\Gamma(-\ell_{23}^1)R_{r_{12},r_{23}}(\ell_2) \\ &\times \begin{pmatrix} e^{-\frac{r_{12}-r_{23}}{2b^2}}\sin\pi\ell_{12}^3 + e^{\frac{r_{12}-r_{23}}{2b^2}}\sin\pi\ell_{23}^1 & e^{\frac{r_{12}+r_{23}}{2b^2}}\sin 2\pi\ell_2 \\ e^{-\frac{r_{12}+r_{23}}{2b^2}}\sin 2\pi\ell_2 & e^{\frac{r_{12}-r_{23}}{2b^2}}\sin\pi\ell_{12}^3 + e^{-\frac{r_{12}-r_{23}}{2b^2}}\sin\pi\ell_{23}^1 \end{pmatrix} . \end{aligned} \tag{3.19} $$

3.3 Check of the geometrical limit

Let me now compute the geometrical limit of the H_3^+ three-point function in order to compare it with the prediction of subsection 2.1. This amounts to taking the level k to infinity (equivalently $b = (k-2)^{-\frac{1}{2}} \to 0$), while keeping the spins ℓ_i fixed, and the boundary parameters r_{ij} fixed and equal to a common value r. Let me perform this limit on the explicit expression for the boundary three-point structure constant (3.13),

$$ \begin{aligned} C_\lambda &= \frac{4}{i\pi^2}\left(\frac{\mu_L}{b^2}\right)^{-\frac{\ell_{123}+2}{2}} \frac{\Gamma_b(-b\ell_{13}^2)\Gamma_b(-b\ell_{12}^3)\Gamma_b(-b(\ell_{123}+2))\Gamma_b(Q+b\ell_{23}^1)}{\Gamma_b(Q)\prod_{i=1}^3\Gamma_b(-b(2\ell_i+1))} \\ &\times \sum_{\sigma=\pm} e^{\lambda\left(\frac{r_{31}}{2b^2}+i\sigma\frac{\pi}{2}\ell_{13}^2\right)} \frac{S_b(\frac{1}{2b}+i\sigma\frac{r_{12}+r_{23}}{2\pi b}-b\ell_2)S_b(i\sigma\frac{r_{23}-r_{12}}{2\pi b}-b\ell_2)}{S_b(\frac{1}{2b}+i\sigma\frac{r_{23}+r_{31}}{2\pi b}+b(\ell_3+1))S_b(i\sigma\frac{r_{23}-r_{31}}{2\pi b}+b(\ell_3+1))} \int dp \\ &\frac{S_b(\frac{1}{2b}+i\sigma\frac{r_{23}+r_{31}}{2\pi b}-b\ell_1+bp)S_b(i\sigma\frac{r_{23}-r_{31}}{2\pi b}-b\ell_1+bp)S_b(-b\ell_2+bp)S_b(Q+b\ell_2+bp)}{\prod_\pm S_b(Q+b(\ell_3^\pm-\ell_1)+bp)S_b(\frac{1}{2b}+i\sigma\frac{r_{12}+r_{23}}{2\pi b}+b+bp)S_b(Q+i\sigma\frac{r_{23}-r_{12}}{2\pi b}+bp)} . \end{aligned} \tag{3.20} $$

Limits of C_λ and Ω_3. The behaviour of the special function S_b as $b \to 0$ is given in eqs. (A.6, A.7). The argument of the function S_b must behave in certain ways for the limit to exist. In the geometrical limit, the spins ℓ_i and brane parameters r_{ij} are kept fixed. This allows C_λ to have a well-defined limit only provided all brane parameters are equal, as was anticipated on more heuristic grounds in subsection 2.1. Calling r this common parameter, and neglecting some numerical prefactors, the limit is found by direct calculation to be

$$ C_\lambda\left(\ell_1 \underset{r}{|} \ell_2 \underset{r}{|} \ell_3 \underset{r}{|}\right) \underset{b\to 0}{\sim} (\cosh r)^{\sum \ell_i + 2}\, e^{\lambda\frac{r}{2b^2}}\, C^0 , \tag{3.21} $$

where the constant C^0, which depends only on the spins ℓ_1, ℓ_2, ℓ_3, is

$$C^0 = \frac{\Gamma(-\ell_{13}^2)\Gamma(-\ell_{12}^3)\Gamma(-\ell_{123}-1)}{\prod_{i=1}^3 \Gamma(-2\ell_i - 1)} \frac{\Gamma(-\ell_2)}{\Gamma(\ell_3+1)} \cos\tfrac{\pi}{2}\ell_{13}^2 \times I \,, \tag{3.22}$$

$$I \equiv \int dp\, \frac{\Gamma(-p)\Gamma(\ell_1-\ell_2-p)\Gamma(-\ell_1+p)\Gamma(-\ell_3+p)\Gamma(\ell_3+1+p)}{\Gamma(-\ell_1-\ell_2+p)} \,. \tag{3.23}$$

Now insert this into the three-point function Ω_3, eq. (2.14), and use formula (A.9) to get the simple result

$$\Omega_3 \underset{b\to 0}{\sim} |t_{12}|^{\ell_{12}^3}|t_{13}|^{\ell_{13}^2}|t_{23}|^{\ell_{23}^1}(\cosh r)^{\sum \ell_i + 2} C^0 \,. \tag{3.24}$$

The dependences on r and t_i therefore agree with the geometrical three-point function Ω_3^{geom} eq. (2.5).

Calculation of C^0. It remains to explicitly compute the integral I. Inserting $1 = i\int_{i\mathbb{R}} dp'\ \delta(ip - ip')$ and $\delta(ip-ip') = \int_0^\infty \frac{dz}{z}\ z^{p+p'}$ yields

$$\begin{aligned} I &= \int_0^\infty \frac{dz}{z} \int_{i\mathbb{R}} dp\, dp'\ z^{p+p'} \frac{\Gamma(-\ell_1+p)\Gamma(-\ell_3+p)}{\Gamma(-\ell_1-\ell_2+p)} \Gamma(-p)\Gamma(\ell_1-\ell_2-p')\Gamma(\ell_3+1+p') \\ &= \Gamma(\ell_{13}^2+1)\frac{\Gamma(-\ell_1)\Gamma(-\ell_3)}{\Gamma(-\ell_1-\ell_2)} \int_0^\infty \frac{dz}{z}(1+z)^{-\ell_{13}^2-1} F(-\ell_1,-\ell_3,-\ell_1-\ell_2,-z) \,. \end{aligned} \tag{3.25}$$

This can be integrated with the help of the formula (A.10), yielding

$$I = \Gamma(-\ell_1)\Gamma(-\ell_3)\Gamma(\ell_3+1)\frac{\Gamma(\ell_{13}^2+1)\Gamma(-\frac{1}{2}\ell_{12}^3+1)\Gamma(-\frac{1}{2}\ell_{23}^1)}{\Gamma(1-\ell_{12}^3)\Gamma(\frac{1}{2}\ell_{13}^2+1)\Gamma(-\frac{1}{2}\ell_{123})} \,. \tag{3.26}$$

It is now easy to compute C^0 and compare it with the result C^{geom} (2.9) of the geometrical calculation,

$$C^0 = N_1 \prod_{i=1}^3 \left(N_2^{\ell_i} \frac{\Gamma(-\ell_i)}{\Gamma(-\ell_i - \frac{1}{2})} \right) C^{\text{geom}} \,, \tag{3.27}$$

where N_1, N_2 are some normalization constants. (Such constants have been neglected in the computation.) Therefore, the $b \to 0$ limit of the exact boundary three-point function agrees with the geometrical boundary three point function, up to an overall renormalization and a renormalization of the vertex operators.

4. Relation with fusing matrix elements

This section is devoted to computing certain fusing matrix elements of the H_3^+ model, and relating them to the boundary three-point function. In the case of Liouville theory, the determination of the fusing matrix was used for finding the boundary three-point function [7]. In the present case of the H_3^+ model, the boundary three-point function is already known, and its relation with the fusing matrix can be deduced from the explicit formula. Apart from testing the validity of general ideas on the structure of conformal field theories, the exercise may help address questions like: Are AdS_2 D-branes the only continuous, maximally symmetric D-branes in H_3^+? How do Euclidean AdS_2 D-branes in H_3^+ compare with Minkowskian AdS_2 D-branes in AdS_3? Tentative answers will be given in the conclusion.

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4.1 An H_3^+ fusing matrix

The fusing matrix of the H_3^+ model can be defined as the linear transformation between bases of s- and t-channel four-point conformal blocks. These four-point conformal blocks are supposed to be completely determined by the symmetry of the model. I will however not try to rigorously define them. Rather, I will adopt the more functional approach of using the H_3^+-Liouville relation for deriving s- and t- channel decompositions of the boundary four-point function. I will call the objects appearing in these decompositions conformal blocks, and compute the corresponding fusing matrix. This approach will be justified a posteriori by the relation between the resulting fusing matrix elements with the boundary three-point function. However, this relation will only involve some particular combinations of fusing matrix elements; a full understanding of the H_3^+ conformal blocks and fusing matrix is left for future work.

I will however need one important insight from the general definition of conformal blocks based on symmetries of the model: namely, that in the H_3^+ model the conformal blocks and fusing matrix are expected to depend on the boundary parameters r_{ij}. This is because the symmetry transformations of the fields (2.10) do themselves depend on r_{ij}. (Like these symmetry transformations, the blocks and fusing matrix should be invariant under shifts $r_{ij} \to r_{ij} + r_0$.) This contrasts with the situation in say Liouville theory [7], where boundary parameters are purely dynamical quantities which affect neither the conformal blocks nor the fusing matrix.

Functional definition of the conformal blocks and fusing matrix. Consider the ν-basis boundary four-point function

$$\tilde{\Omega}_4 = \left\langle {}_{r_{41}}\Psi^{\ell_1}(\nu_1|w_1)_{r_{12}}\Psi^{\ell_2}(\nu_2|w_2)_{r_{23}}\Psi^{\ell_3}(\nu_3|w_3)_{r_{34}}\Psi^{\ell_4}(\nu_4|w_4)_{r_{41}} \right\rangle . \tag{4.1}$$

The s-channel and t-channel four-point conformal blocks

$$\mathcal{G}^{\ell_s}_{\lambda_{12}\lambda_{34}}\left(\ell_1 \underset{r_{12}}{|} \ell_2 \underset{r_{23}}{|} \ell_3 \underset{r_{34}}{|} \ell_4 \underset{r_{41}}{|} \,\middle|\, \nu_i \,\middle|\, w_i\right) \quad , \quad \mathcal{G}^{\ell_t}_{\lambda_{23}\lambda_{14}}\left(\ell_1 \underset{r_{12}}{|} \ell_2 \underset{r_{23}}{|} \ell_3 \underset{r_{34}}{|} \ell_4 \underset{r_{41}}{|} \,\middle|\, \nu_i \,\middle|\, w_i\right) , \tag{4.2}$$

are defined as the quantities appearing in the s-channel and t-channel decompositions of $\tilde{\Omega}_4$,

$$\begin{aligned} \tilde{\Omega}_4 &= \sum_{\lambda_{12},\lambda_{34}} \int_{-\frac{1}{2}+i\mathbb{R}} d\ell_s \; \left(R^L(\beta_s)\right)^{-1} C_{\lambda_{12}}\left(\ell_1 \underset{r_{12}}{|} \ell_2 \underset{r_{23}}{|} \ell_s \underset{r_{41}}{|}\right) C_{\lambda_{34}}\left(\ell_3 \underset{r_{34}}{|} \ell_4 \underset{r_{41}}{|} \ell_s \underset{r_{23}}{|}\right) \mathcal{G}^{\ell_s}_{\lambda_{12}\lambda_{34}}, &(4.3)\\ &= \sum_{\lambda_{23},\lambda_{41}} \int_{-\frac{1}{2}+i\mathbb{R}} d\ell_t \; \left(R^L(\beta_t)\right)^{-1} C_{\lambda_{23}}\left(\ell_2 \underset{r_{23}}{|} \ell_3 \underset{r_{34}}{|} \ell_t \underset{r_{12}}{|}\right) C_{\lambda_{41}}\left(\ell_4 \underset{r_{41}}{|} \ell_1 \underset{r_{12}}{|} \ell_t \underset{r_{34}}{|}\right) \mathcal{G}^{\ell_t}_{\lambda_{23}\lambda_{41}}, &(4.4) \end{aligned}$$

which otherwise involve the three-point structure constant C_λ and the ν-basis reflection coefficient $R^L_{\frac{r_{23}}{2\pi b}-\frac{i}{4b}\mathrm{sgn}(\nu_1+\nu_2),\frac{r_{41}}{2\pi b}+\frac{i}{4b}\mathrm{sgn}(\nu_1+\nu_2)}\left(b(\ell_s+1)+\frac{1}{2b}\right)$ eq. (3.17).

The fusing matrix is defined as realizing the change of basis between s- and t-channel blocks,

$$\mathcal{G}^{\ell_s}_{\lambda_{12}\lambda_{34}} = \sum_{\lambda_{23},\lambda_{41}} \int_{-\frac{1}{2}+i\mathbb{R}} d\ell_t \; F^{\ell_s\ell_t}_{\lambda_{12}\lambda_{34}\lambda_{23}\lambda_{14}} \begin{bmatrix} \ell_3 & r_{23} & \ell_2 \\ r_{34} & & r_{12} \\ \ell_4 & r_{41} & \ell_1 \end{bmatrix} \; \mathcal{G}^{\ell_t}_{\lambda_{23}\lambda_{41}} . \tag{4.5}$$

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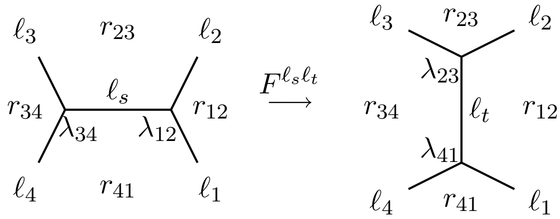
The conformal blocks and their fusion transformation will be depicted as



. (4.6)

H_3^+ conformal blocks from Liouville conformal blocks. The H_3^+ boundary four-point function can be written in terms of a Liouville boundary six-point function as [1]:

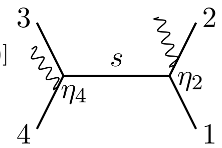
$$\tilde{\Omega}_4 = \delta(\textstyle\sum \nu_i)|\textstyle\sum \nu_i w_i| \left| \dfrac{y_{12} \prod_{i<i'} w_{ii'}}{\prod_{a=1,2} \prod_i (y_a - w_i)} \right|^{\frac{1}{2b^2}} \left\langle \prod_{i=1}^{4} B^{\beta_i}(w_i) \;\; B^{-\frac{1}{2b}}(y_1) B^{-\frac{1}{2b}}(y_2) \right\rangle , \tag{4.7}$$

where $\beta_i = b(\ell_i + 1) + \frac{1}{2b}$ as before, the Liouville boundary parameter is still given by eq. (3.3), and y_1, y_2 are still defined as the zeroes of a function $\varphi(y)$ (3.2). The idea is now to decompose the Liouville six-point function in terms of Liouville structure constants and conformal blocks, out of which the H_3^+ structure constants C_λ and conformal blocks should be reconstructed. The details of the decomposition are quite sensitive on signs of the isospin variables ν_i, which determine the positions of the Liouville degenerate fields $B^{-\frac{1}{2b}}(y_1), B^{-\frac{1}{2b}}(y_2)$ on the worldsheet boundary. (In some cases, the degenerate fields can even live in the bulk.) Such subtleties would be very relevant to a rigorous definition of the conformal blocks; but here I will neglect them and assume

$$(\mathrm{sgn}\nu_1, \mathrm{sgn}\nu_2, \mathrm{sgn}\nu_3, \mathrm{sgn}\nu_4) = (+, -, -, -) \quad \Rightarrow \quad w_2 < y_1 < w_3 < y_2 < w_4 \ . \tag{4.8}$$

Now I claim that, in this regime, s-channel blocks can be built in terms of Liouville blocks as

$$\mathcal{G}^{\ell_s}_{\lambda_{12}\lambda_{34}} = N^{+}_{r_{41},r_{23}}(\ell_s) \sum_{\eta_2,\eta_4} T^{[+1(2)s]}_{\lambda_{12},\eta_2} T^{[+s3(4)]}_{\lambda_{34},\eta_4}$$

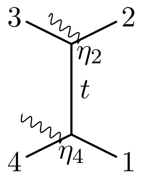


, (4.9)

where, in the diagrammatic representation of the standard six-point Liouville blocks, the wiggly lines are the degenerate fields, whose fusion channels are labelled $\eta = \pm$ like in the four-point Liouville blocks of eq. (3.4), and the solid lines are the generic fields with momenta $\beta_1, \beta_2, \beta_3, \beta_4, \beta_s$. (The prefactors in eq. (4.7) are implicitly included in the Liouville blocks.) (Remember that $N^{\sigma}_{r,r'}(\ell)$ was defined in (3.16), and $T^{[\mathrm{sgn}\nu_i]}_{\lambda,\eta}$ in (2.18).)

The proof that such s-channel blocks do indeed satisfy eq. (4.3) is straightforward, given the relation (3.12) between Liouville and H_3^+ boundary three-point structure constants. It is of course also possible to find t-channel blocks satisfying eq. (4.4),

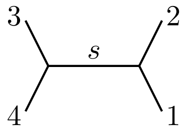
$$\mathcal{G}^{\ell_t}_{\lambda_{23}\lambda_{41}} = N^{-}_{r_{12},r_{34}}(\ell_t) \sum_{\eta_2,\eta_4} T^{[+t(2)3]}_{\lambda_{23},\eta_2} T^{[+1t(4)]}_{\lambda_{41},\eta_4}$$



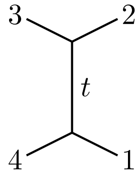
(4.10)

Let me now derive the fusing matrix which relates these s- and t-channel blocks.

H_3^+ fusing matrix from Liouville fusing matrix. The relation between the Liouville conformal blocks appearing in the H_3^+ s- and t-channel conformal blocks is given by the Liouville fusing matrix, which is defined by [8]



$$= \int_{\frac{Q}{2}+i\mathbb{R}} d\beta_t \; F^L_{\beta_s\beta_t}\begin{bmatrix} \beta_3 & \beta_2 \\ \beta_4 & \beta_1 \end{bmatrix} \tag{4.11}$$



Applying this relation to the Liouville blocks appearing in the formulas for H_3^+ four-point blocks (4.9) and (4.10) yields an H_3^+ fusing matrix satisfying eq. (4.5):

$$F^{\ell_s\ell_t}_{\lambda_{12}\lambda_{34}\lambda_{23}\lambda_{14}}\begin{bmatrix} \ell_3 & r_{23} & \ell_2 \\ r_{34} & & r_{12} \\ \ell_4 & r_{41} & \ell_1 \end{bmatrix} = \frac{N^+_{r_{41},r_{23}}(\ell_s)}{N^-_{r_{12},r_{34}}(\ell_t)} \tag{4.12}$$
$$\times \sum_{\eta_2,\eta_4} T^{[+1(2)s]}_{\lambda_{12},\eta_2} T^{[+s3(4)]}_{\lambda_{34},\eta_4} F^L_{\beta_s\beta_t}\begin{bmatrix} \beta_3 & \beta_2 - \frac{\eta_2}{2b} \\ \beta_4 - \frac{\eta_4}{2b} & \beta_1 \end{bmatrix} \left(T^{-1}\right)^{[+t(2)3]}_{\eta_2,\lambda_{23}} \left(T^{-1}\right)^{[+1t(4)]}_{\eta_4,\lambda_{41}} \; .$$

Notice that the four Liouville fusing matrix elements appearing in this formula are not all independent, but can be related to any two of them via linear equations whose coefficients are products of Gamma functions. (See appendix A.3.)

It can actually be proved that this fusing matrix satisfies a Pentagon equation, but this is outside the scope of this article. In general conformal field theories, the Pentagon equation is the structural reason for the existence of a relation between the fusing matrix and the boundary three-point function. Here I will however derive such a relation by direct calculation.

4.2 Discrete representations of $\widetilde{SL}(2,\mathbb{R})$

This subsection is a technical interlude devoted to the definition and study of the discrete representations of $\widetilde{SL}(2,\mathbb{R})$. There may seem to be no physical motivation for studying such representations in the context of the H_3^+ model, whose spectrum is purely continuous. However, it will turn out that discrete representations play a crucial role in the relation between the fusing matrix and the boundary three-point function.[5]

Discrete representations and discrete fields. There are two series of discrete representations, called D^+_ℓ and D^-_ℓ. A representation $D^\pm_\ell$ is defined as having a state which is annihilated by the generator $J^\mp$ of the $s\ell_2$ Lie algebra, whose commutation relations and quadratic Casimir operator are

$$[J^3, J^\pm] = \pm J^\pm, \;\; [J^+, J^-] = -2J^3, \quad C = -(J^3)^2 + \tfrac{1}{2}(J^+J^- + J^-J^+) \; . \tag{4.13}$$

[5]Note that by focusing on the $\widetilde{SL}(2,\mathbb{R})$ horizontal subgroup I am still ignoring the rest of the infinite-dimensional symmetry group of the model. Representations of $\widetilde{SL}(2,\mathbb{R})$ can however easily be extended to highest-weight representations of the full symmetry group. Anyway, since discrete representations are absent from the spectrum, their structure will be of no importance in the following. Only formal properties like the allowed values of the spins will be needed.

The eigenvalues of C are labelled in terms of the spin ℓ as $C = -\ell(\ell+1)$, and the eigenvalues of J^3 are called m. The J^--annihilated state of a D^+_ℓ representation can have either $m = \ell + 1$ or $m = -\ell$. In the case $\ell \in \frac{1}{2}\mathbb{Z}$, such a state must have $m > 0$, otherwise a J^+-annihilated state appears at $J^3 = -m$, and the representation is finite-dimensional instead of being discrete. In the case of generic ℓ however, both D^+_ℓ representations should be accepted, but distinguishing them will not matter in the following. I will also ignore the special case $\ell \in \frac{1}{2}\mathbb{Z}$. Note however that discrete representations of $SL(2,\mathbb{R})$ must have $\ell \in \frac{1}{2}\mathbb{Z}$, whereas discrete representations of the universal cover $\widetilde{SL}(2,\mathbb{R})$ exist for all $\ell \in \mathbb{C}$.

A field $\Psi^\ell(t)$ belonging to the $D^\pm_\ell$ representation can be analytically continued to the half-plane $U^\pm \equiv \{\pm\Im t > 0\}$ [13]. So if $\Psi^{\ell_2}(t_2) \in D^\sigma_{\ell_2}$ with $\sigma = \pm$, then the t-basis three-point function Ω_3 (2.14) must be analytic in $t_2 \in U^\pm$. This constrains its behaviours near $t_2 = t_1$ and $t_2 = t_3$. For instance, near $t_2 = t_1$ the relevant factors of Ω_3 behave as $\Omega_3 \propto |t_{12}|^{\ell^3_{12}} e^{\frac{k-2}{2} r_{12}\mathrm{sgn}t_{12}} C_{-\mathrm{sgn}t_{12}}$, which has an analytic continuation to $t_2 \in U^\sigma$ provided $e^{i\pi\sigma\ell^3_{12}} e^{-(k-2)r_{12}} C_- = C_+$. Together with the condition $e^{i\pi\sigma\ell^1_{23}} e^{(k-2)r_{23}} C_+ = C_-$ from $t_2 \sim t_3$, this is equivalent to

$$\ell_2 \in \sigma\frac{k-2}{2\pi i}(r_{12} - r_{23}) + \mathbb{Z}\,, \tag{4.14}$$

$$C_\lambda = \lambda^n \; e^{\frac{\lambda}{2}\left[i\pi\sigma(\ell_1-\ell_3) - \frac{k-2}{2}(r_{12}+r_{23})\right]} C_0\,, \tag{4.15}$$

where C_0 is a λ-independent constant, and $n \in \{0, 1\}$ is the parity of the element of $\mathbb{Z}$ above. The condition on ℓ_2 depends only on the field ${}_{r_{12}}\Psi^{\ell_2}(t_2)_{r_{23}}$ and not on the other fields in the three-point function, and it is the condition for that field to be discrete.

The interesting feature of discrete representations is therefore the disappearance of the multiplicity λ in the boundary interactions: a three-point function involving a discrete representation is determined in terms of only one structure constant C_0, instead of $C_\pm$ in the generic case.

Discrete ν-basis fields. Since the investigation of the fusing matrix in H^+_3 heavily relied on the ν-basis, it will be necessary to understand how fields transforming in discrete representations behave in the ν-basis. The analyticity of discrete fields for $t \in U^\pm$ translates into corresponding ν-basis fields $\Psi^\ell(\nu) = |\nu|^{\ell+1} \int_{\mathbb{R}} dt\; e^{i\nu t}\Psi^\ell(t)$ vanishing for $\pm\nu > 0$. How does this simplify the coefficients $T^{[\pm i(j)k]}_{\lambda,\eta}$ eq. (2.22)–(2.23), which enter the formula for the fusing matrix? The coefficients $T^{[\sigma 1(2)3]}_{\lambda,\eta}$ are defined for $\mathrm{sgn}\nu_2 = -\sigma$, and the explicit formula shows

$$\ell_2 \in -\sigma\frac{k-2}{2\pi i}(r_{12} - r_{23}) + \mathbb{Z} \quad \Rightarrow \quad T^{[\sigma 1(2)3]}_{\lambda,-} = 0\ . \tag{4.16}$$

What if it is the third field in Ω_3 which belongs to a discrete representation $D^\sigma_{\ell_3}$? Then similarly $T^{[\sigma 12(3)]}_{\lambda,-} = 0$, and the relation (2.25) yields

$$\ell_3 \in \sigma\frac{k-2}{2\pi i}(r_{23} - r_{31}) + \mathbb{Z} \quad \Rightarrow \quad \frac{T^{[\sigma 1(2)3]}_{\lambda,+}}{T^{[\sigma 1(2)3]}_{\lambda,-}} = \frac{M^{(23)1}_{++}}{M^{(23)1}_{-+}}\,, \tag{4.17}$$

so that $T^{[\sigma 1(2)3]}_{\lambda,+}$ must have the same λ-dependence as $T^{[\sigma 1(2)3]}_{\lambda,-}$. Finally, what if it is the first field in Ω_3 which now belongs to $D^{\sigma}_{\ell_1}$? Just use the explicit formulas for $T^{[\sigma 1(2)3]}_{\lambda,\eta}$ to read off how they behave under $1 \leftrightarrow 3$, and deduce from the previous case

$$\ell_1 \in -\sigma \frac{k-2}{2\pi i}(r_{31} - r_{12}) + \mathbb{Z} \quad \Rightarrow \quad \frac{T^{[\sigma 1(2)3]}_{\lambda,+}}{T^{[\sigma 1(2)3]}_{\lambda,-}} = \frac{M^{(21)3}_{++}}{M^{(21)3}_{-+}} \frac{\sin\left(\pi \ell_2 - i\sigma \frac{r_{23}-r_{12}}{2b^2}\right)}{\sin\left(\pi \ell_2 + i\sigma \frac{r_{23}-r_{12}}{2b^2}\right)} \,. \tag{4.18}$$

4.3 Relation fusing matrix — boundary three-point function

The case of Liouville theory. Let me begin with recalling the form of this relation in Liouville theory. On the one hand this will be useful in the derivation of the H_3^+ relation, on the other hand this will illustrate what type of relation should be expected.

The Liouville boundary three-point function (3.6) is related to the Liouville fusing matrix (4.11) by [7]

$$C^L \left(\beta_1 \underset{s_{12}}{|} \beta_2 \underset{s_{23}}{|} \beta_3 \underset{s_{31}}{|} \right) = R^L_{s_{31},s_{12}}(\beta_1) \frac{g^L_{s_{31},s_{12}}(\beta_1)}{g^L_{s_{12},s_{23}}(\beta_2) g^L_{s_{23},s_{31}}(\beta_3)} F^L_{\frac{Q}{2}+is_{23},\beta_1} \begin{bmatrix} \beta_3 & \beta_2 \\ \frac{Q}{2}+is_{31} & \frac{Q}{2}+is_{12} \end{bmatrix} , \tag{4.19}$$

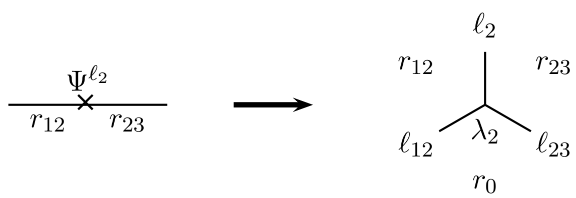
where the function $g^L_{s,s'}(\beta)$, which may be seen as a sort of square root of the reflection coefficient (3.10) and satisfies $g^L_{s,s'}(\beta) = R^L_{s,s'}(Q-\beta) g^L_{s,s'}(Q-\beta)$, is

$$g^L_{s,s'}(\beta) = \mu_L^{\frac{1}{2}b^{-1}\beta} \frac{\Gamma_b(Q)\Gamma_b(Q-2\beta)\Gamma_b(Q+2is)\Gamma_b(Q-2is')}{\prod_{\pm\pm} \Gamma_b(Q-\beta \pm is \pm is')} \,. \tag{4.20}$$

The basic idea, which is originally due to Cardy [14], is therefore to associate some momenta $\beta_{ij} = \frac{Q}{2} + is_{ij}$ to the boundary conditions s_{ij}. These momenta are then used as inputs in the fusing matrix [15].

Peculiarities of the H_3^+ model. Unlike Liouville theory, the H_3^+ model does not a priori conform to the assumptions which would make these ideas work. In particular, the $SL(2,\mathbb{C})$ representations appearing in the bulk spectrum are labelled by their sole spin, whereas the $\widetilde{SL}(2,\mathbb{R})$ representations appearing in the boundary spectrum are labelled by a spin and an extra continuous parameter $\alpha = r - r'$ depending on the boundary parameters r, r'. Associating bulk spins to the boundary conditions may be useful to some extent for understanding the moduli space of D-branes in H_3^+ [16], but the inputs in the H_3^+ fusing matrix rather need to be pairs (ℓ, α) as in the boundary spectrum.

Another feature of the H_3^+ case is the presence of a multiplicity index λ in the three-point structure constant C_λ, and of four corresponding indices in the fusing matrix. The generic expectation [17, 18] is that such multiplicities should also appear as indices of the boundary fields themselves. There should indeed be a correspondence between boundary fields and three-point vertices:



(4.21)

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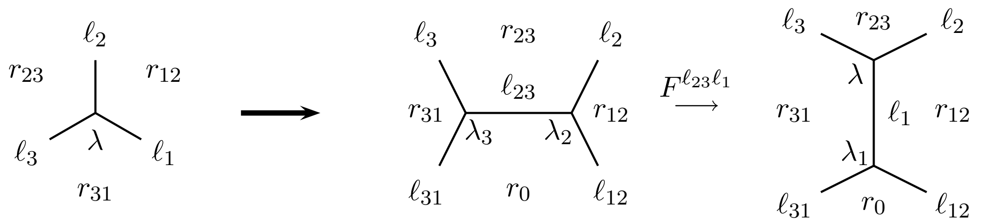
Which spins ℓ_{12}, ℓ_{23} should correspond to the boundary conditions r_{12}, r_{23}? What should r_0 and λ_2 be? The idea proposed here is to choose ℓ_{ij} as discrete spins, which would eliminate the index λ_2 as explained in the previous subsection, and determine r_0. The relation between boundary three-point function and the fusing matrix will then be of the type:



, (4.22)

where the dependence of the fusing matrix $F^{\ell_{23}\ell_1}$ on $\lambda_1, \lambda_2, \lambda_3$ is trivial thanks to the spins ℓ_{ij} being discrete.

Derivation of the relation by direct calculation. I will not seek further guidance from general structural ideas, but rather from the explicit formulas. Namely, I will use the relations between the H_3^+ and Liouville three-point structure constants (3.12), then between the Liouville structure constant and fusing matrix (4.19), and finally between the Liouville and H_3^+ fusing matrices (4.12). Specifically, start with

$$C_\lambda = \sum_{\eta=\pm} C^L_\sigma(\beta_1|\beta_2 - \tfrac{\eta}{2b}|\beta_3)\ C^L_\sigma(\beta_2| - \tfrac{1}{2b}|Q-\beta_2+\tfrac{\eta}{2b})\ \left(T^{-1}\right)^{[\sigma 1(2)3]}_{\eta,\lambda}\ , \tag{4.23}$$

and insert the expression for $C^L_\sigma(\beta_1|\beta_2 - \frac{\eta}{2b}|\beta_3)$ in terms of the Liouville fusing matrix in the case $\sigma = +$,

$$\begin{aligned} C_\lambda =\ & R^L_{\frac{r_{31}}{2\pi b}+\frac{i}{4b},\frac{r_{12}}{2\pi b}-\frac{i}{4b}}(\beta_1) \frac{g^L_{\frac{r_{31}}{2\pi b}+\frac{i}{4b},\frac{r_{12}}{2\pi b}-\frac{i}{4b}}(\beta_1)}{g^L_{\frac{r_{23}}{2\pi b}-\frac{i}{4b},\frac{r_{31}}{2\pi b}+\frac{i}{4b}}(\beta_3)} \sum_{\eta=\pm} \frac{C^L_+(\beta_2|-\frac{1}{2b}|Q-\beta_2+\frac{\eta}{2b})}{g^L_{\frac{r_{12}}{2\pi b}-\frac{i}{4b},\frac{r_{23}}{2\pi b}-\frac{i}{4b}}(\beta_2-\frac{\eta}{2b})} \\ & \times \left(T^{-1}\right)^{[+1(2)3]}_{\eta,\lambda} F^L_{\frac{Q}{2}-\frac{r_{23}}{2\pi ib}+\frac{1}{4b},\beta_1}\begin{bmatrix} \beta_3 & \beta_2-\frac{\eta}{2b} \\ \frac{Q}{2}-\frac{r_{31}}{2\pi ib}-\frac{1}{4b} & \frac{Q}{2}-\frac{r_{12}}{2\pi ib}+\frac{1}{4b}\end{bmatrix}\ . \end{aligned} \tag{4.24}$$

This combination $\sum_{\eta=\pm}$ of two F^L matrices should be compared to the combination appearing in the following rewriting of the H_3^+ fusing matrix (4.12), where I use the property $T^{[+12(3)]} = T^{[-1(3)2]}$:

$$\begin{aligned} & \sum_{\lambda_1} T^{[-,12,(31),1]}_{\lambda_1,\eta_0} F^{\ell_{23}\ell_1}_{\lambda_2\lambda_3\lambda\lambda_1}\begin{bmatrix} \ell_3 & r_{23} & \ell_2 \\ r_{31} & & r_{12} \\ \ell_{31} & r_0 & \ell_{12}\end{bmatrix} \\ & = \frac{N^+_{r_0,r_{23}}(\ell_{23})}{N^-_{r_{12},r_{31}}(\ell_1)} T^{[-,23,(31),3]}_{\lambda_3,\eta_0} \sum_{\eta=\pm} T^{[+,12,(2),23]}_{\lambda_2,\eta} \left(T^{-1}\right)^{[+1(2)3]}_{\eta,\lambda} F^L_{\beta_{23},\beta_1}\begin{bmatrix} \beta_3 & \beta_2-\frac{\eta}{2b} \\ \beta_{31}-\frac{\eta_0}{2b} & \beta_{12}\end{bmatrix}\ . \end{aligned} \tag{4.25}$$

The F^L fusing matrices which appear in the last two equations are equal provided their arguments are identical modulo reflection $\beta \to Q - \beta$. This is the case if one assumes $\eta_0 = +$ and $\beta_{ij} = b(\ell_{ij}+1) + \frac{1}{2b}$ with

$$\ell_{ij} = -\frac{1}{2} - \frac{r_{ij}}{2\pi i b^2} + \frac{1}{4b^2}\ . \tag{4.26}$$

This relation between spins and boundary parameters agrees with the one proposed in [16]. However, the idea is now to interpret the corresponding representations as discrete representations. This is possible if the relation $\ell_{ij} \in -\frac{k-2}{2\pi i}(r_{ij} - r_0) + \mathbb{Z}$ is obeyed. And this relation indeed holds provided the following assumption is made:

$$r_0 = i\pi \left(\frac{1}{2} - b^2 \right) \ . \tag{4.27}$$

Then, according to the formulas (4.17) and (4.18), the factors $T^{[-,12,(31),1]}_{\lambda_1,\eta_0}$, $T^{[-,23,(31),3]}_{\lambda_3,\eta_0}$ and $T^{[+,12,(2),23]}_{\lambda_2,\eta}$ simplify (without vanishing), in the sense that their λ and η-dependences disentangle. In particular, the λ_2-dependence in eq. (4.25) can be rewritten as a prefactor, outside the sum $\sum_{\eta=\pm}$.

Test and results. Now that the parameters r_0, ℓ_{ij} are fixed, comes the test: are the combinations of two F^L-matrices in (4.24) and (4.25) proportional up to an overall factor? Direct calculations (which use eq. (4.17)) indeed show that they are, thanks to the following identity, valid for any $\sigma = \pm$:

$$\frac{1}{R^L_{\frac{r_{12}}{2\pi b}-\sigma\frac{i}{4b},\frac{r_{23}}{2\pi b}+\sigma\frac{i}{4b}}(\beta_2)} \frac{g^L_{\frac{r_{12}}{2\pi b}-\sigma\frac{i}{4b},\frac{r_{23}}{2\pi b}-\sigma\frac{i}{4b}}(Q-\beta_2-\frac{1}{2b})}{g^L_{\frac{r_{12}}{2\pi b}-\sigma\frac{i}{4b},\frac{r_{23}}{2\pi b}-\sigma\frac{i}{4b}}(\beta_2-\frac{1}{2b})} = \frac{T^{[\sigma,12,(2),23]}_{\lambda_2,+}}{T^{[\sigma,12,(2),23]}_{\lambda_2,-}} \ . \tag{4.28}$$

It is then possible to define coefficients of the type

$$g^\lambda_{r,r'}(\ell) = e^{-\lambda\left[i\frac{\pi}{2}(\ell+1)+\frac{r+r'}{4b^2}\right]} g^0_{r,r'}(\ell) \, , \tag{4.29}$$

where $g^0_{r,r'}(\ell)$ is λ-independent, such that for all $\lambda_2, \lambda_3 = \pm$

$$C_\lambda \left(\ell_1 \underset{r_{12}}{|} \ell_2 \underset{r_{23}}{|} \ell_3 \underset{r_{31}}{|} \right) = R_{r_{31},r_{12}}(\ell_1) \sum_{\lambda_1=\pm} \frac{g^{\lambda_1}_{r_{31},r_{12}}(\ell_1)}{g^{\lambda_2}_{r_{12},r_{23}}(\ell_2) g^{\lambda_3}_{r_{23},r_{31}}(\ell_3)} F^{\ell_{23}\ell_1}_{\lambda_2\lambda_3\lambda\lambda_1} \begin{bmatrix} \ell_3 & r_{23} & \ell_2 \\ r_{31} & & r_{12} \\ \ell_{31} & r_0 & \ell_{12} \end{bmatrix} \tag{4.30}$$

This is the sought-after expression for the boundary three-point function in terms of fusing matrix elements, which depend on the particular arguments r_0 and ℓ_{ij} defined above. This result can be rewritten in terms of a "partly discrete fusing matrix" $\tilde{F}$ such that

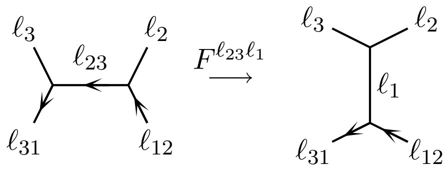
$$C_\lambda \left(\ell_1 \underset{r_{12}}{|} \ell_2 \underset{r_{23}}{|} \ell_3 \underset{r_{31}}{|} \right) = R_{r_{31},r_{12}}(\ell_1) \frac{g^0_{r_{31},r_{12}}(\ell_1)}{g^0_{r_{12},r_{23}}(\ell_2) g^0_{r_{23},r_{31}}(\ell_3)} \tilde{F}^{\ell_{23}\ell_1}_\lambda \begin{bmatrix} \ell_3 & \ell_2 \\ \ell_{31} & \ell_{12} \end{bmatrix} \ . \tag{4.31}$$

In this notation, the H_3^+ result becomes very similar to the Liouville result (4.19).

Representation-theoretic discussion. Let me now check that the use of discrete representations in the fusing matrix, as suggested by the above calculations, is actually compatible with the algebraic properties of these representations. Unfortunately, the fusion products of vertex operators with $\widetilde{SL}(2,\mathbb{R})$ symmetry, and even the tensor products of $\widetilde{SL}(2,\mathbb{R})$ representations, are apparently unknown. However, some features can be extrapolated from the known $SL(2,\mathbb{R})$ representations, where tensor products of the type

$D^+ \otimes D^-$ are expected to yield continuous representations (and possibly discrete ones), whereas tensor products $D^+ \otimes D^+$ or $D^- \otimes D^-$ only yield discrete representations. These statements should also hold for fusion products of $\widetilde{SL}(2,\mathbb{R})$ representations.

It is therefore important to determine whether the discrete representations of spins ℓ_{ij} (4.26) belong to the D^+ or to the D^- series. According to the rule (4.14), and taking good care of the orientation of the worldsheet boundary, the discrete representations are found to be $D^+_{\ell_{12}}$, $D^-_{\ell_{31}}$ and $D^\pm_{\ell_{23}}$. The sign in $D^\pm_{\ell_{23}}$ depends on a choice of orientation, as can be seen in the following oriented depiction of the fusing matrix (4.22),



. (4.32)

In this picture, incoming arrows denote D^+ representations, outgoing arrows denote D^- representations, and lines without arrows denote C (Continuous) representations. The vertices involving discrete representations are all of the  type, and they therefore correspond to non-vanishing $D^+ \otimes D^- \to C$ intertwiners.

5. Conclusions and speculations

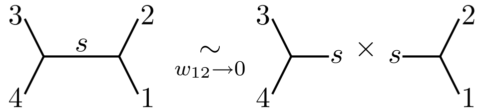
Another limit of the boundary three-point function. The geometrical (or minisuperspace) limit of the boundary three-point function has provided a non-trivial check of the exact formula, see subsection 3.3. In this limit, the brane parameters r_{12}, r_{23}, r_{31} are kept fixed, and the limit then exists only provided they are all equal. It is however interesting to consider another $b \to 0$ limit, where the quantities $R_{ij} \equiv \frac{r_{ij}}{2\pi b^2}$ are kept fixed. This limit no longer requires them to be equal, and can be explicitly computed from eqs. (2.14) and (3.20):

$$\begin{aligned}\Omega_3 \;\sim\; & |t_{12}|^{\ell^3_{12}} |t_{13}|^{\ell^2_{13}} |t_{23}|^{\ell^1_{23}} e^{\pi R_{12}\mathrm{sgn} t_{12} + \pi R_{23} \mathrm{sgn} t_{23} + \pi R_{31} \mathrm{sgn} t_{31}} \qquad (5.1)\\ & \frac{\Gamma(-\ell^2_{13})\Gamma(-\ell^3_{12})\Gamma(-\ell_{123}-1)}{\prod_{i=1}^3 \Gamma(-2\ell_i - 1)} \sum_{\sigma=\pm} e^{\pi\left(R_{31} + \frac{1}{2} i\sigma\ell^2_{13}\right)\mathrm{sgn} t_{12}t_{23}t_{31}} \frac{\Gamma\left(i\sigma[R_{23}-R_{12}] - \ell_2\right)}{\Gamma\left(i\sigma[R_{23}-R_{31}] + \ell_3 + 1\right)} \\ & \int dp\; \frac{\Gamma(i\sigma[R_{23}-R_{31}]+p)\Gamma(\ell_1-\ell_2+p)\Gamma(i\sigma[R_{12}-R_{23}]-\ell_1-p)\prod_\pm \Gamma(-\ell_3^\pm - p)}{\Gamma(-\ell_1-\ell_2-p)} \,.\end{aligned}$$

This limit has an analog in the case of D-branes in SU(2): the Alekseev-Recknagel-Schomerus limit where maximally symmetric D-branes become fuzzy spheres [19]. In the rational SU(2) theory, the algebra of boundary fields on a given D-brane then becomes a finite-dimensional matrix algebra, with the size of the matrices depending on the boundary parameter. In the present H_3^+ case, the algebra of boundary fields is infinite-dimensional, and may have an interpretation as the algebra of functions on a non-compact, non-commutative AdS_2 manifold. The above limit of Ω_3 would then describe the product in this algebra, whose noncommutativity ultimately comes from the lack of worldsheet parity invariance of the H_3^+ model with boundary.

Towards the Minkowskian theory. Solving the H_3^+ model may be seen as a step in the study of string theory in the Minkowskian AdS_3. On the one hand, this theory is expected to be technically more complicated due to the presence of discrete and spectrally flowed representations in the spectrum [20], in addition to the purely continuous spectrum of the H_3^+ model. On the other hand, the formal structure of the theory is probably more conventional, since the symmetry algebra safely factorizes into left- and right-movers.

Let me explain why the formalism of the present article may be well-suited to studying strings in the Minkowskian AdS_3. The conventionality of the formal structure of that theory suggests that AdS_3 four-point conformal blocks could be defined using the usual factorization assumption. This assumption is that in the limit $w_{12} \to 0$, where two fields come close together on the worldsheet, the s-channel four-point blocks should factorize into products of three-point blocks:



(5.2)

(It can be seen that the H_3^+ blocks defined in section 4.1 do not obey this assumption.) Now, this assumption would lead to s-channel blocks being singular at $\nu_s \equiv \nu_1 + \nu_2 = 0$, simply because the three-point blocks themselves are. This $\nu_s = 0$ singularity takes very characteristic forms when discrete and spectrally flowed representations propagate in the s-channel. As was recalled in section 4.2, an s-channel field in a discrete representation would indeed vanish for either $\nu_s < 0$ or $\nu_s > 0$. I now add that a spectrally flowed field would be a distribution supported at $\nu_s = 0$, as can be deduced from [21]. Therefore, ν-basis blocks permit an easy characterization of continous, discrete and spectrally flowed s-channel modes, based on their behaviour near $\nu_s = 0$.

New D-branes in H_3^+? The relation between the boundary three-point function and the fusing matrix (4.30) relies on associating certain boundary fields to the boundary conditions of the model. However, the boundary conditions only have a real parameter r, and they are associated a the set of discrete boundary fields, which are far from exhausting the full space of boundary fields ${}_r\Psi^\ell_{r'}$ parametrized by their spin ℓ and by $\alpha = r - r'$. Can fusing matrices with generic entries (ℓ, α) be interpreted as three-point structure constants on new maximally symmetric D-branes? If not, why do the discrete representations, and only them, give rise to D-branes in H_3^+?

Acknowledgments

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A. Some useful formulas

A.1 Special functions Γ_b and S_b

The special functions Γ_b and S_b usually appear in the study of Liouville theory at parameter $b > 0$ and background charge $Q = b + b^{-1}$. I use the same conventions as [9], where some more details can be found. The following definitions are valid for $0 < \Re x < Q$:

$$\log\Gamma_b(x) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-xt} - e^{-Qt/2}}{(1-e^{-bt})(1-e^{-t/b})} - \frac{(Q/2-x)^2}{2}e^{-t} - \frac{Q/2-x}{t}\right] , \tag{A.1}$$

$$\log S_b = \int_0^\infty \frac{dt}{t}\left[\frac{\sinh(\frac{Q}{2}-x)t}{2\sinh(\frac{bt}{2})\sinh(\frac{t}{2b})} - \frac{(Q-2x)}{t}\right] . \tag{A.2}$$

These functions, which are related by $S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}$, can be extended to meromorphic functions on the complex plane thanks to the shift equations

$$\Gamma_b(x+b) = \frac{\sqrt{2\pi}b^{bx-\frac{1}{2}}}{\Gamma(bx)}\Gamma_b(x) \quad , \quad \Gamma_b(x+1/b) = \frac{\sqrt{2\pi}b^{-\frac{x}{b}+\frac{1}{2}}}{\Gamma(x/b)}\Gamma_b(x) \tag{A.3}$$

$$S_b(x+b) = 2\sin(\pi bx)S_b(x) \quad , \quad S_b(x+1/b) = 2\sin(\pi x/b)S_b(x) \tag{A.4}$$

Using the integral representations for the special functions, one can study their behaviour for $b \to 0$ while keeping the quantities x, y fixed:

$$\Gamma_b(bx) \to (2\pi b^3)^{\frac{1}{2}(x-\frac{1}{2})}\Gamma(x) \quad , \quad \Gamma_b(Q-bx) \to \left(\frac{b}{2\pi}\right)^{\frac{1}{2}(x-\frac{1}{2})} , \tag{A.5}$$

$$S_b(bx) \to (2\pi b^2)^{x-\frac{1}{2}}\Gamma(x) \quad , \quad S_b(\frac{1}{2b}+bx) \to 2^{x-\frac{1}{2}} , \tag{A.6}$$

$$|\Re y| < \frac{1}{2} \Rightarrow S_b(\tfrac{1}{2b}+bx+\tfrac{1}{b}y) \quad \to \quad \left(\frac{\cos\pi y}{2}\right)^{\frac{1}{2}-x}\exp -\frac{1}{b^2}\int_0^\infty \frac{dt}{t}\left[\frac{\sinh 2yt}{2t\sinh t} - \frac{y}{t}\right] . \tag{A.7}$$

A.2 Miscellaneous

The following integral [4], which should be understood as a distribution, appears in eq. (2.8).

$$\int_{\mathbb{R}} dy\ e^{i\theta y}|y|^\alpha = \frac{2}{|\theta|^{\alpha+1}}\Gamma(\alpha+1)\sin\tfrac{\pi}{2}\alpha . \tag{A.8}$$

The following identity, which is valid for three arbitrary real numbers t_1, t_2, t_3, is applied to isospins in eqs. (2.14) and (3.24).

$$\mathrm{sgn} t_{12}t_{23}t_{31} + \mathrm{sgn} t_{12} + \mathrm{sgn} t_{23} + \mathrm{sgn} t_{31} = 0 . \tag{A.9}$$

An integral formula from [10] (7.512) is used in eq. (3.25):

$$\int_0^1 dx\ x^{\alpha-\gamma}(1-x)^{\gamma-\beta-1}F(\alpha,\beta,\gamma,x) = \frac{\Gamma(1+\frac{1}{2}\alpha)\Gamma(\gamma)\Gamma(\alpha-\gamma+1)\Gamma(\gamma-\beta-\frac{1}{2}\alpha)}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}\alpha+1-\beta)\Gamma(\gamma-\frac{1}{2}\alpha)} \tag{A.10}$$

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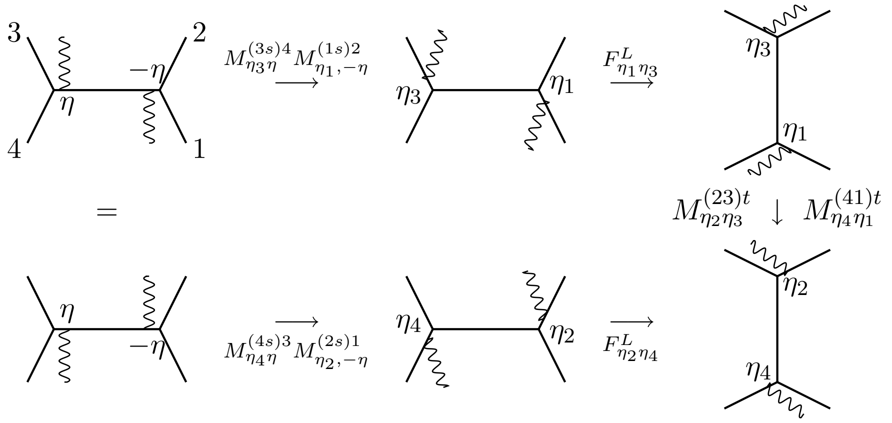
A.3 Linear equations for certain Liouville fusing matrices

Let me derive linear relations involving the fusing matrices $F^L_{\eta_1\eta_3} \equiv F^L_{\beta_s\beta_t}\left[\begin{smallmatrix} \beta_3-\frac{\eta_3}{2b} & \beta_2 \\ \beta_4 & \beta_1-\frac{\eta_1}{2b}\end{smallmatrix}\right]$ and $F^L_{\eta_2\eta_4} \equiv F^L_{\beta_s\beta_t}\left[\begin{smallmatrix} \beta_3 & \beta_2-\frac{\eta_2}{2b} \\ \beta_4-\frac{\eta_4}{2b} & \beta_1\end{smallmatrix}\right]$, where $\eta_i = \pm$ are signs. I will use a sequence of Liouville fusing transformations, including some degenerate ones whose matrix elements are the $M^{(ij)k}_{\eta\eta'}$ defined in eq. (2.26):



(A.11)

Each choice of $\eta = \pm$ yields a formula for the four matrix elements $F^L_{\eta_2\eta_4}$ in terms of $F^L_{\eta_1\eta_3}$:

$$\forall \eta = \pm, \qquad F^L_{\eta_2\eta_4} = \sum_{\eta_1,\eta_3} M^{(23)t}_{\eta_2\eta_3} M^{(41)t}_{\eta_4\eta_1} F^L_{\eta_1\eta_3} \frac{M^{(3s)4}_{\eta_3,-\eta}}{M^{(4s)3}_{\eta_4,-\eta}} \frac{M^{(1s)2}_{\eta_1\eta}}{M^{(2s)1}_{\eta_2\eta}} \,. \tag{A.12}$$

Using both choices $\eta = \pm$, one can eliminate $F^L_{\eta_2\eta_4}$ and find the following rank two system of four equations for $F^L_{\eta_1\eta_3}$, where $J \equiv b^{-1}(\beta - \frac{Q}{2})$:

$$\begin{aligned}\forall \eta_2,\eta_4, \qquad & \sum_{\eta_1,\eta_3} \frac{\prod_\pm \Gamma(\frac{1}{2} \pm J_s + \eta_3 J_3 - \eta_4 J_4) \prod_\pm \Gamma(\frac{1}{2} \pm J_s + \eta_1 J_1 - \eta_2 J_2)}{\prod_\pm \Gamma(\frac{1}{2} \pm J_t + \eta_3 J_3 - \eta_2 J_2) \prod_\pm \Gamma(\frac{1}{2} \pm J_t + \eta_1 J_1 - \eta_4 J_4)} \\ & \times \sin\pi(\eta_2 J_2 + \eta_3 J_3 - \eta_1 J_1 - \eta_4 J_4) \quad F^L_{\eta_1\eta_3} = 0 \,. \end{aligned} \tag{A.13}$$

References

[1] K. Hosomichi and S. Ribault, *Solution of the H_3^+ model on a disc*, *JHEP* **01** (2007) 057 [hep-th/0610117].

[2] J. Teschner, *On structure constants and fusion rules in the* SL(2, ℂ)/ SU(2) *WZNW model*, *Nucl. Phys.* **B 546** (1999) 390 [hep-th/9712256].

[3] K. Hosomichi, *A correspondence between H_3^+ WZW and Liouville theories on discs*, *Nucl. Phys.* **171** *(Proc. Suppl.)* (2007) 284 [hep-th/0701260].

[4] B. Ponsot, V. Schomerus and J. Teschner, *Branes in the euclidean AdS_3*, *JHEP* **02** (2002) 016 [hep-th/0112198].

[5] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, *Boundary Liouville field theory. I: boundary state and boundary two-point function*, hep-th/0001012.

[6] J. Teschner, *Remarks on Liouville theory with boundary*, hep-th/0009138.

[7] B. Ponsot and J. Teschner, *Boundary Liouville field theory: boundary three point function*, *Nucl. Phys.* **B 622** (2002) 309 [hep-th/0110244].

[8] J. Teschner, *Liouville theory revisited*, *Class. and Quant. Grav.* **18** (2001) R153 [hep-th/0104158].

[9] B. Ponsot, *Recent progresses on Liouville field theory*, *Int. J. Mod. Phys.* **A 19S2** (2004) 311 [hep-th/0301193].

[10] I. Gradshteyn and I. Ryzhik, *Table of integrals, series and products*, Academic Press (1965).

[11] S. Ribault and J. Teschner, *H_3^+ WZNW correlators from Liouville theory*, *JHEP* **06** (2005) 014 [hep-th/0502048].

[12] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, *Nucl. Phys.* **B 241** (1984) 333.

[13] I.M. Gelfand, M.I. Graev and N.Y. Vilenkin, *Generalized functions, volume* 5, Academic Press (1966).

[14] J.L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, *Nucl. Phys.* **B 324** (1989) 581.

[15] I. Runkel, *Boundary structure constants for the A-series Virasoro minimal models*, *Nucl. Phys.* **B 549** (1999) 563 [hep-th/9811178].

[16] S. Ribault, *Discrete D-branes in AdS_3 and in the 2D black hole*, *JHEP* **08** (2006) 015 [hep-th/0512238].

[17] R.E. Behrend, P.A. Pearce, V.B. Petkova and J.-B. Zuber, *Boundary conditions in rational conformal field theories*, *Nucl. Phys.* **B 570** (2000) 525 [*Nucl. Phys.* **B 579** (2000) 707] [hep-th/9908036].

[18] J. Fuchs, I. Runkel and C. Schweigert, *TFT construction of RCFT correlators. IV: structure constants and correlation functions*, *Nucl. Phys.* **B 715** (2005) 539 [hep-th/0412290].

[19] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Non-commutative world-volume geometries: branes on* SU(2) *and fuzzy spheres*, *JHEP* **09** (1999) 023 [hep-th/9908040].

[20] J.M. Maldacena and H. Ooguri, *Strings in AdS_3 and* $\mathrm{SL}(2,\mathbb{R})$ *WZW model. I*, *J. Math. Phys.* **42** (2001) 2929 [hep-th/0001053].

[21] S. Ribault, *Knizhnik-Zamolodchikov equations and spectral flow in AdS_3 string theory*, *JHEP* **09** (2005) 045 [hep-th/0507114].